Onset of plane layer magnetoconvection at low Ekman number

Kélig Aujogue, Alban Pothérat, and Binod Sreenivasan

Citation: Physics of Fluids 27, 106602 (2015); doi: 10.1063/1.4934532
View online: http://dx.doi.org/10.1063/1.4934532
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/27/10?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in

Successive elimination of shear layers by a hierarchy of constraints in inviscid spherical-shell flows

Dynamo action in an annular array of helical vortices

Driven inertial waves in spherical Couette flow
Chaos 16, 041105 (2006); 10.1063/1.2390555

Anisotropic evolution of small isolated vortices within the core of the Earth

Computational Aspects of Geodynamo Simulations
Comput. Sci. Eng. 2, 61 (2000); 10.1109/5992.841797
Onset of plane layer magnetoconvection at low Ekman number

Kélig Aujogue,1,a) Alban Pothérat,1 and Binod Sreenivasan2
1Applied Mathematics Research Centre, Coventry University, Priory Street, Coventry CV1 5FB, United Kingdom
2Centre for Earth Sciences, Indian Institute of Science, Bangalore 560012, India
(Received 7 July 2014; accepted 8 October 2015; published online 26 October 2015)

We study the onset of magnetoconvection between two infinite horizontal planes subject to a vertical magnetic field aligned with background rotation. In order to gain insight into the convection taking place in the Earth’s tangent cylinder, we target regimes of asymptotically strong rotation. The critical Rayleigh number \( Ra_c \) and critical wavenumber \( a_c \) are computed numerically by solving the linear stability problem in a systematic way, with either stress-free or no-slip kinematic boundary conditions. A parametric study is conducted, varying the Ekman number \( E \) (ratio of viscous to Coriolis forces) and the Elsasser number \( \Lambda \) (ratio of the Lorentz force to the Coriolis force). \( E \) is varied from \( 10^{-9} \) to \( 10^{-2} \) and \( \Lambda \) from \( 10^{-3} \) to 1. For a wide range of thermal and magnetic Prandtl numbers, our results verify and confirm previous experimental and theoretical results showing the existence of two distinct unstable modes at low values of \( E \)--one being controlled by the magnetic field, the other being controlled by viscosity (often called the viscous mode). It is shown that oscillatory onset does not occur in the range of parameters we are interested in. Asymptotic scalings for the onset of these modes are numerically confirmed and their domain of validity is precisely quantified. We show that with no-slip boundary conditions, the asymptotic behavior is reached for \( E < 10^{-6} \) and establish a map in the \((E, \Lambda)\) plane. We distinguish regions where convection sets in either through the magnetic mode or through the viscous mode. Our analysis gives the regime in which the transition between magnetic and viscous modes may be observed. We also show that within the asymptotic regime, the role played by the kinematic boundary conditions is minimal. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4934532]

I. INTRODUCTION

In this paper, we analyse the onset of plane-layer convection governed by the interplay between the magnetic (Lorentz), buoyancy, and Coriolis forces, to obtain an insight into how convection in the tangent cylinder (TC) region of the Earth’s liquid core is driven. This region is bounded by the Earth’s solid inner core at its bottom, the mantle at its top, and by an imaginary cylinder tangent to the solid inner core and parallel to the Earth’s rotation axis. Intense convection, compositional and thermal, is believed to take place in this region, affecting the structure of the magnetic field near the poles.1 The Earth’s self-generated magnetic field is thought to affect the structure of convective cells in the TC, producing strong anticyclonic polar vortices that show up in the secular variation of the geomagnetic field.2 The aim of our study is to find out whether onset of convection is sensitive to the Lorentz force in the regime of strong rotation that characterises the Earth.

Previous work on plane rotating magnetoconvection has been motivated either by geophysical or engineering applications involving liquid metals.3–9 A number of geophysically motivated studies focused on the dynamics outside the TC: an early study10 derived theoretical scalings for the critical Rayleigh number and wave number at the onset of convection as a function of the magnetic field

\[ \text{a)} \text{aujoguek@uni.coventry.ac.uk} \]
intensity and magnitude of the Coriolis force. Other studies \cite{4,11,12} showed experimentally and theoretically that, in this region convection and rotation generated tall columns parallel to the rotation axis. A recent study investigated the role of a dipolar magnetic field in enhancing helicity in convection columns, \cite{13} which can explain subcritical behavior in rapidly rotating dynamos. These studies, however, do not consider the particularity of the TC, which, though imaginary, acts somewhat as a physical boundary because the presence of the solid inner core makes overcoming the Taylor-Proudman constraint more difficult. When convection does set in, motions vary strongly along \( z \) as heat and composition flux have a substantial component in the \( z \)-direction. Due to the large aspect ratio of the TC, the curvature of the top and the bottom boundaries is not expected to play a lead role, at least at the onset of convection. On these grounds, a simple plane geometry is expected to provide a fair, albeit simplistic, representation of the TC. In this geometry, it was theorised \cite{14} and experimentally observed \cite{15} that the convection could set off through an instability either of a magnetic or a viscous mode, depending on the values of the Ekman number (ratio of viscous to Coriolis forces) and of the Elsasser number (ratio of Lorentz to Coriolis forces). While the magnetic mode has a low horizontal wavenumber, the viscous mode is characterised by thin structures of high horizontal wavenumber parallel to the rotation axis. One would expect that convective flows driven by these two mechanisms differ significantly. These studies showed that transition between these modes resulted in a brutal change in the wavelength of the observed convective pattern, but concerned only large values of \( E (>10^{-5}) \). Such values may be too far from the asymptotic regimes relevant to the Earth’s core (\( E \sim 10^{-15} \)) to be applicable to it.

There have been experimental studies applicable to the dynamics of the TC for \( E = 10^{-4} \), \cite{16,17} but only the viscous mode of convection could be observed. The link between plane layer magnetocoction and convection in the Earth’s TC was first established by linear onset calculations as well as numerical simulations of the geodynamo, \cite{1,2} where substantial thickening of buoyant plumes under the effect of the magnetic field was noted, albeit at values of \( E \) down to \( 10^{-4} \) only. Crucially, these studies showed that non-axisymmetric, Earth-like polar vortices are obtained only through the action of the magnetic field.

In this study, we look at plane-layer magnetocoction at values of \( E \) low enough to reach asymptotic regimes. \cite{18,19} We precisely determine the range of parameters in which these asymptotics become accurate. Although actual regimes of the TC remain beyond the reach of this analysis, asymptotic scalings are relevant to it. In the same spirit, we shall characterise the consequence of using either a no-slip boundary condition or its less computationally demanding stress-free counterpart on these regimes.

The paper is organised as follows: Section II introduces the governing equations and the numerical method to solve them. The results of the stability analysis are discussed in Section III considering non-oscillatory unstable modes only. Scalings for the asymptotic regimes are given in terms of \( E \) and \( \Lambda \). The precise conditions for the onset of overstable (oscillatory) convection are established in Section IV. Relevance to the Earth is discussed in Section V.

\section*{II. GOVERNING EQUATIONS AND NUMERICAL METHOD}

\subsection*{A. Governing equations}

We consider an incompressible fluid (kinematic viscosity \( \nu \), thermal diffusivity \( \kappa \), magnetic diffusivity \( \eta \), density \( \rho \), thermal expansion coefficient \( \alpha \) ) confined between two differentially heated infinite horizontal plane boundaries, separated by a distance \( d \). The temperature difference between them is \( \Delta T \) and the uniform gravity \( g \) points vertically downward. The flow rotates at a speed \( \Omega \) about the vertical axis \( z \) and is permeated by a uniform vertical magnetic field \( B = B_0 \hat{\epsilon}_z \). Figure 1 illustrates our geometry.

The flow is governed by the full incompressible magnetohydrodynamic (MHD) equations in the Boussinesq approximation. Normalising lengths by \( d \), the velocity by \( \eta/d \), the pressure by \( \rho_0 \eta \Omega \), the magnetic field by \( B_0 \), the time by \( d^2/\eta \), the temperature by \( \Delta T \), and the rotation speed by \( \Omega \), the equations can be written in the following non-dimensional form:

\begin{equation}
\frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot \mathbf{u} + \frac{\mathbf{u} \cdot \nabla \mathbf{u}}{\nu} + 2 \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} - \Omega \times \mathbf{u} + R_i T \hat{\mathbf{z}} + \Lambda \left( (\nabla \times \mathbf{B}) \times \mathbf{B} \right) + E \nabla^2 \mathbf{u},
\end{equation}
FIG. 1. Schematic of the plane layer magnetoconvection problem.

\[ \frac{\partial B}{\partial t} = \nabla \times (u \times B) + \nabla^2 B, \] \(\text{Eq. (2)}\)

\[ \frac{\partial T}{\partial t} + (u \cdot \nabla)T = Pr^{-1} \nabla^2 T, \] \(\text{Eq. (3)}\)

\[ \nabla \cdot u = 0, \] \(\text{Eq. (4)}\)

\[ \nabla \cdot B = 0, \] \(\text{Eq. (5)}\)

where \(B\) is the total magnetic field. The system is controlled by five non-dimensional parameters—the Ekman number \(E = \nu/\Omega d^2\), a modified Rayleigh number \(Ra = g\alpha \Delta T d/\eta \Omega\), the Elsasser number \(\Lambda = B^2/\mu_0 \eta \rho \Omega\), the Prandtl number \(Pr = \nu/\kappa\), and the magnetic Prandtl number \(Pm = \nu/\eta\). We apply three different kinds of boundary conditions: stress-free magnetic (SFM, conditions (6)–(9) below), no-slip magnetic (NSM, conditions (8)–(10) below), and no-slip conducting (NSC, conditions (9)–(11) below). These are given, respectively, for \(z = \pm 1/2\) as

\[ u_z = 0 \quad \text{(impermeability)}, \] \(\text{Eq. (6)}\)

\[ \frac{d^2 u_z}{dz^2} = 0 \quad \text{(stress-free)}, \] \(\text{Eq. (7)}\)

\[ (\nabla \times B)_e = 0 \quad \text{(electrically insulating)}, \] \(\text{Eq. (8)}\)

\[ T(z = -1/2) = 1, \ T(z = +1/2) = 0, \] \(\text{Eq. (9)}\)

\[ u = 0 \quad \text{(no-slip)}, \] \(\text{Eq. (10)}\)

\[ \frac{d(\nabla \times B)}{dz} e_z = 0 \quad \text{(electrically conducting)}. \] \(\text{Eq. (11)}\)

For all sets of boundary conditions, the system has a simple basic state solution whose linear stability we are interested in. The problem’s invariance in the \(x\) and \(y\) directions allows us to decompose perturbations from the basic state as

\[ f(z, t) = f_0 + \hat{f}(z)e^{i(a_\perp r_x + \sigma_\parallel t)}, \] where \(r_\perp = (x, y)\), \(a\) is the wave number and \(\sigma\) is the growth rate. Following Sreenivasan and Jones,\(^1\) we shall only seek the shape of the unstable modes, not their growth rate. However, at this stage we shall not exclude oscillatory modes in order to determine the precise conditions of their appearance. The perturbation equations are given by

\[ E(D^2 - a^2 + Pm^{-1} \sigma_\parallel) \hat{\omega}_z + 2D \hat{u}_z + \Lambda D \hat{j}_z = 0, \] \(\text{Eq. (12)}\)

\[ E((D^2 - a^2)^2 + Pm^{-1} \sigma_\parallel) \hat{u}_z - 2D \hat{\omega}_z + \Lambda(D^2 - a^2) D \hat{b}_z - a^2 R a \hat{T}_\parallel = 0, \] \(\text{Eq. (13)}\)

\[ (D^2 - a^2 + \sigma_\parallel) \hat{b}_z + D \hat{u}_z = 0, \] \(\text{Eq. (14)}\)

\[ (D^2 - a^2 + \sigma_\parallel) \hat{j}_z + D \hat{\omega}_z = 0, \] \(\text{Eq. (15)}\)

\[ (Pm Pr^{-1}(D^2 - a^2) + \sigma_\parallel) \hat{T}_\parallel + \hat{u}_z = 0. \] \(\text{Eq. (16)}\)
Here $D$ is the derivative along $z$, $\sigma_i$ is the imaginary part of $\sigma$, $\omega_z$, $\dot{u}_z$, $\dot{j}_z$, and $\dot{b}_z$ are the $z$-components of the vorticity, velocity, electric current, and magnetic field perturbations, and $\hat{T}'$ is the temperature perturbation. The non-dimensional wavenumber is denoted by $a = \| \mathbf{a} \|$. Eq. (12) is obtained from $\nabla \times (1) \cdot \mathbf{e}_z$, Eq. (13) from $\nabla \times [\nabla \times (1)] \cdot \mathbf{e}_z$, Eq. (14) from $\nabla \times (2) \cdot \mathbf{e}_z$, and Eq. (16) follows from Eq. (3). Boundary conditions (6)–(10) take the form

$$D^2 \hat{u}_z = D \hat{\omega}_z = \dot{j}_z = \dot{T}' = 0 \quad \text{for} \quad z = \pm 1/2 \quad (\text{SFM}),$$

$$D \hat{u}_z = \hat{\omega}_z = \dot{j}_z = \hat{T}' = 0 \quad \text{for} \quad z = \pm 1/2 \quad (\text{NSM}),$$

$$D \hat{u}_z = \hat{\omega}_z = D \dot{j}_z = \hat{T}' = 0 \quad \text{for} \quad z = \pm 1/2 \quad (\text{NSC}).$$

The problem becomes a generalized eigenvalue problem of the form $A(\sigma_i) \chi = RaBX$, where $\sigma_i$ is treated as a parameter. The critical Rayleigh number for the onset of convection $Ra_c(\sigma_i)$ is found as an eigenvalue for any given $a$ and minimised over $a$ as in Ref. 14. When the marginal mode is not oscillatory, the critical Rayleigh number is found by setting $\sigma_i = 0$ in Eqs. (12)–(16). In this case, with help of a formal transformation of $\hat{T}'$ into $T_m' = Prm^{-1} \hat{T}'$, the solution is made independent of the magnetic and thermal diffusivities. The results presented thereafter on the stability of non-oscillatory modes therefore extend to arbitrary values of $Prm$ and $Pr$. The non-oscillatory modes are also referred to as steady modes.\textsuperscript{20}

B. Numerical method

Eqs. (12)–(16) were solved numerically using a spectral collocation method based on Chebyshev polynomials.\textsuperscript{21} In the no-slip case, a boundary layer of thickness $\delta = 2 \sqrt{E \pi}$ develops along the walls,\textsuperscript{22} and therefore we ensured that at least 3 collocation points were in it. Some convergence tests were performed to ensure that the resolution is adequate. The results for $\sigma_i = 0$ are presented in Figure 2, where we varied the number of collocation points $N$ between 5 and 3000. In the SFM case, the tests were performed for $\Lambda = 1$, $E = 10^{-9}$, and $a = 3.149$. NSM and NSC conditions were tested with $\Lambda = 1$, $E = 10^{-7}$, and $a = 3.333$. We chose these parameters to ensure a good convergence at the lowest $E$ we investigated. We looked at the value of the error $\epsilon$ on $Ra_c$ relative to its value obtained for $N = 3000$. For both types of boundary conditions, $N > 100$ gives a small relative error. On the basis of this test, the results presented in Sec. III have been obtained with $N = 600$ for the SFM case, $N = 1200$ for the NSM case, and $N = 1000$ for the NSC case.

We performed a parametric study with $a = [1, 1500]$, $E = [10^{-9}, 10^{-2}]$, and $\Lambda = [10^{-3}, 2]$ for SFM and $a = [1, 1500]$, $E = [10^{-8}, 10^{-2}]$, and $\Lambda = [10^{-3}, 2]$ for NSM and NSC.
FIG. 3. Variation of \( R_a \) with \( \alpha \) in the case of SFM boundaries. The continuous blue curve’s input parameters are \( E = 10^{-8} \) and \( \Lambda = 1 \). The dotted green curve’s input parameters are \( E = 10^{-8} \) and \( \Lambda = 10^{-1} - 10^{-3} \). The dashed red curve’s input parameters are \( E = 10^{-5} - 10^{-7} \) and \( \Lambda = 1 \).

III. STABILITY OF STATIONARY MODES (\( \sigma_t = 0 \))

A. General properties

In Figure 3, we show the typical behavior of the critical Rayleigh number \( R_a \) with respect to the wave number \( \alpha \). The continuous blue curve corresponds to \( E = 10^{-8} \) and \( \Lambda = 1 \). The dotted green curves are for \( E = 10^{-8} \) and \( \Lambda = 10^{-1} - 10^{-3} \) and the dashed red curves for \( E = 10^{-5} - 10^{-7} \) at \( \Lambda = 1 \). For each case, we note three specific values for \( R_a \). The first is a minimum occurring at low \( \alpha \); its position and value are mostly controlled by \( \Lambda \) and not by \( E \). As such, it is referred to as the magnetic mode which we shall denote by \( (R_a^m, a^m) \), where \( R_a^m \) is the magnetic critical Rayleigh number and \( a^m \) is the magnetic critical wave number. The second is a local minimum at relatively high \( \alpha \), its position and value depending essentially on \( E \). We shall refer to it as the viscous mode \( (R_a^v, a^v) \), where \( R_a^v \) is the viscous critical Rayleigh number and \( a^v \) is the viscous critical wavenumber. Both these modes were first identified by Chandrasekhar.\(^{14}\) The third feature is a local maximum located between the two previous modes. We call this the intermediate maximum and denote it by \( (R_a^{int}, a^{int}) \). The corresponding mode is always more stable than both the magnetic and the viscous mode and does not reflect any mechanism driving convection. At low \( E \), the value of \( R_a^{int} \) is several orders of magnitude higher than \( R_a^v \) and \( R_a^m \). The intermediate maximum gives a measure of how much of a separation exists between magnetically controlled modes and modes controlled by viscosity.

B. Scalings for the critical wavenumber and Rayleigh number

In Figures 4(a) and 4(b), we show the variations of \( R_a \) and \( a_c \) with \( E \) at \( \Lambda = [0.1, 0, 3, 1, 2] \) for the viscous mode, magnetic mode, and for the intermediate maximum identified in \( \Lambda \) with SFM boundary conditions. We note two important results in the limit of \( E \rightarrow 0 \). First, the scalings obtained for the viscous modes reproduce the classical results of non-magnetic convection, that is, \( R_a^v \propto 22.3E^{-1/3} \) and \( a^v \propto 1.65E^{-1/3} \); and for the intermediate maximum, \( R_a^{int} \propto E^{-1/2} \) and \( a^{int} \propto E^{-1/4} \). Second, at low \( E \), convection is initiated by the instability of the magnetic mode. On the other hand, when \( E \) increases at a fixed value of \( \Lambda \), \( R_a^c \) decreases while \( R_a^m \) remains constant, so that a crossover value \( E_c(\Lambda) \) exists beyond which the viscous mode is more unstable and triggers the onset of convection. Before this point is reached, the clear separation between magnetic and viscous modes progressively starts disappearing. Ultimately, the intermediate maximum merges into the magnetic mode, at which point both disappear, for \( E = E_P < E_c(\Lambda) \).

In Figures 5(a) and 5(b), we report the variations of \( R_a^m, R_a^{int}, R_a^v, a^m, a^{int}, \) and \( a^v \) with \( \Lambda \) for \( E = 10^{-8} \) and \( E = 10^{-7} \). Values of the Elsasser number \( \Lambda \) below 10 have been considered but particular attention has been given to \( \Lambda \) around 1, which is relevant to the Earth’s core. For higher values
FIG. 4. Variation of critical Rayleigh number $R a_c$ (a) and critical wavenumber $a_c$ (b) with Ekman number $E$, for SFM boundary conditions.

of the Elsasser number, Sreenivasan and Jones\(^1\) showed that the Lorentz force had a stabilising effect on the flow so that $R a_m^m(\Lambda)$ increases instead of decreasing as it does for $\Lambda < 1$. Interestingly, we note the existence of an absolute minimum for $R a_c$ at $\Lambda \approx 3.47 \pm 0.08$. The existence of this absolute minimum has been observed in several studies.\(^{10,14,23,24}\) These studies concerned the outer part of the Earth core, and so the imposed magnetic field was proportional to the distance from the rotation axis and azimuthal. Zhang and Schubert\(^25\) attributed the presence of the minimum to the unphysical nature of the imposed magnetic field; however, this minimum is a robust feature of plane layer magnetoconvection problems.

In the limit of $\Lambda \to 0$, we observe that the intermediate maximum scales as $R a_c^{int} \propto \Lambda^{-1/2}$ and $a_c^{int} \propto \Lambda^{1/4}$. For the magnetic mode, on the other hand, $R a_m^m \propto 160 \Lambda^{-3}$ so that the separation between magnetic and viscous modes becomes more and more pronounced as $\Lambda$ increases. Interestingly, $a_m^m$ is practically independent of $\Lambda$ and $E$. The crossover point at which the magnetic mode becomes more unstable than the viscous mode can also be seen.

Figures 6(a), 6(b) and 7(a), 7(b) present the counterparts of Figures 4(a), 4(b) and 5(a), 5(b) for the problem with NSM boundary conditions, while Figures 8(a), 8(b), 9(a), and 9(b) are the corresponding results obtained with NSC boundary conditions. In both cases, the figures indicate that the qualitative behavior of the critical Rayleigh numbers and the critical wavenumbers remains the same as in the configuration with SFM. In particular, the scalings for $R a_c$ and $a_c$ in the limit...
FIG. 5. Variation of critical Rayleigh number $Ra_c$ (a) and critical wavenumber $a_c$ (b) with Elsasser number $\Lambda$, for SFM boundary conditions.

$E \to 0$ and $\Lambda \to 0$ remain valid with $Ra_c^m \propto 19.7E^{-1/3}$ (NSM and NSC), $a_c^m \propto 1.55E^{-1/3}$ (NSM and NSC), $Ra_c^m \propto 150\Lambda^{-1}$ (NSM), and $Ra_c^m \propto 75\Lambda^{-1}$ (NSC). Generally speaking, the destabilising effect of the electromagnetic force is significantly stronger with electrically conducting (NSC) than with insulating walls (NSM). First, the stationary modes set in at a significantly lower value of $Ra$ when the boundaries are conducting. Second, the viscous mode is absent for $\Lambda > 0.3$ and $E \leq 5 \times 10^{-4}$ with NSC, while it is present at $\Lambda = 2$ with NSM. The minimum of $Ra_c^m(\Lambda)$, on the other hand, takes place at about the same value for both types of electrical boundary conditions.

The results obtained with SFM and NSM boundary conditions corroborate the findings of Sreenivasan and Jones\(^1\) that the mechanical boundary conditions at $z = -1/2$ and $z = +1/2$ have little influence on the onset of convection in these limits. One important difference between the two configurations, however, is that convergence towards the asymptotic scalings is significantly slower with NSM boundary conditions than with SFM boundary conditions (with a typical difference of two decades in $E$ and one decade in $\Lambda$). With SFM boundary conditions, at high $E$ and for $\Lambda = 1$, the intermediate maximum merges with the viscous mode rather than with the magnetic mode. This behavior can be expected to take place with NSM boundary conditions too since the wavenumbers of all three modes become closer to each other as $\Lambda$ increases. Our results confirm the relevance of the problem with SFM boundary conditions to the more realistic problem with NSM boundary conditions.
FIG. 6. Variation of critical Rayleigh number $R_{ac}$ (a) and critical wavenumber $a_c$ (b) with Ekman number $E$, for NSM boundary conditions. The small influence of the boundaries comes as a useful feature given that simulations with NSM boundary conditions are considerably more computationally expensive than those with SFM boundary conditions.

C. Parametric study in the $(\Lambda, E)$ space

Figures 10–12 map the mechanisms responsible for the onset of convection in the $(\Lambda, E)$ space. The blue squares represent the area in which only the viscous mode exists. The green triangles characterise the range of parameters where magnetic and viscous modes are present but the most unstable is the viscous one. Finally, the red circles correspond to regimes where both magnetic and viscous modes are present but where the magnetic mode is more unstable than the viscous mode. In these figures, we draw two lines, one marked with dotted red triangles and the other marked with dashed blue triangles. These lines indicate, respectively, the merging of the intermediate maximum with the magnetic mode where $R_{ac}^{m} = R_{ac}^{int}$ and the crossover line where $R_{ac}^{v} = R_{ac}^{int}$. In the limit of $E \rightarrow 0$, these, respectively, obey the scalings

$$\Lambda = 270E,$$

and
FIG. 7. Variation of critical Rayleigh number $R_a$ (a) and critical wavenumber $a_c$ (b) with Elsasser number $\Lambda$, for NSM boundary conditions.

\[ \Lambda = 7.22E^{1/3}. \]  

The exponents in these laws readily follow from the scalings for $R_a m$, $R_a u$, and $R_a u^t$ obtained earlier. These scalings are noted for all types of boundary conditions, and in agreement with Refs. 1 and 20. In the NSM and NSC cases, however, this asymptotic behavior becomes only apparent for $E \sim 10^{-5.5}$ and $E \sim 10^{-4.5}$, respectively. Furthermore, with NSM boundary conditions the transition between magnetic and viscous modes in the non-asymptotic regime still follows a behavior that is qualitatively similar to the asymptotic one, whereas with NSC boundary conditions, the high-$E$ transition strongly departs from the asymptotic transition (see Figure 12). Additionally, consistent with the destabilising effect of the conducting boundaries, the transition between magnetic and viscous modes with NSC boundary conditions, while of the same asymptotic nature as with NSM, takes place at noticeably lower Elsasser number. For NSC conditions,

\[ \Lambda = 3.6E^{1/3}. \]  

The scaling with SFM boundary conditions at moderate $E$ is obtained with NSM boundary conditions at very low $E$. Moreover, the fact that the shape of the $R_a$ curve is independent of the diffusivities $\kappa$ and $\eta$ allows us to mark out the area of parameters investigated in laboratory
FIG. 8. Variation of critical Rayleigh number $Ra_c$ (a) and critical wavenumber $a_c$ (b) with Ekman number $E$, for NSC boundary conditions.

In particular, the experiments of Aurnou and Olson\textsuperscript{16} operate outside the viscous–magnetic transition, whereas the experiments of Nakagawa\textsuperscript{15} capture this transition. In any case, none of these experiments appears to have reached the asymptotic regime of low $E$.

IV. CONDITIONS FOR OSCILLATORY ONSET

The onset of convection occurs through the destabilisation either of the stationary mode ($\sigma_i = 0$) or of an oscillatory mode (overstability). We shall now determine in which range of parameters the stationary mode, which we calculated in Section III, is responsible for the onset of convection. The question of overstability is particularly important with electrically conducting boundaries, since without rotation, these have been shown to favor the onset of convection through the generation of thermal Alfvén waves.\textsuperscript{26}

A. SFM boundary conditions

Chandrasekhar\textsuperscript{14} gives the equations to derive $Ra_c$ for overstable onset for the rotating magnetoconvective case with SFM boundaries. For a given set of parameters ($E$, $\Lambda$, $Pr$, $Pm$), we first
FIG. 9. Variation of critical Rayleigh number $Ra_c$ (a) and critical wavenumber $\alpha_c$ (b) with Elsasser number $\Lambda$, for NSC boundary conditions.

FIG. 10. Characterisation of modes in the $(\Lambda, E)$ space with stress-free insulating (SFM) boundary conditions.
determine the set of values of $a$ for which overstability is possible using Equation (66) in page 210 of Ref. 14. If overstability occurs, we then use Equation (65), page 210 of Ref. 14 to evaluate the corresponding value of the critical Rayleigh number $R_{ac}(E, \Lambda, Pr, Pm, a)$, and minimize this value over the interval spanned by $a$. Finally, we compare this result with the critical Rayleigh number of the stationary case obtained in Sec. III. If the stationary $Ra_c$ is lower, the onset is considered non-oscillatory.

In Figure 13, we show the critical Rayleigh numbers for the most unstable stationary mode and the most unstable overstable mode for selected sets of parameters. This analysis was performed for $E \in [10^{-9}, 10^{-2}]$, $\Lambda \in [10^{-3}, 1]$, $Pr \in [10^{-5}, 1]$, and $Pm \in [10^{-5}, 1]$. It turns out that in this range of parameters, the stationary mode is always the most unstable and therefore determines the onset.

B. NSM and NSC boundary conditions

For these types of boundary conditions, there is no simple analytical way to determine the border between overstable and stationary onsets. Instead, we solve the full system of Equations (12)–(16) with time dependence, varying $\sigma_i$ in the range $[0, 10^{-1}]$. This way, we can numerically determine whether the critical Rayleigh number obtained for $\sigma_i = 0$ is a minimum. When this is the case, the critical mode is obtained when $\sigma_i = 0$ so the onset does not occur through the overstability route. This technique offers a convenient (although not strictly rigorous) way to
determine the overstability condition without incurring the cost of solving the full stability problem for \( \sigma \) as a complex eigenvalue and with \( Ra \) as a parameter.

Figures 14 and 15 show, respectively, the variations of \( Ra_c(\sigma_i) \) around \( \sigma_i = 0 \) for selected values of \( E \), \( \Lambda \), and \( Pm \) for NSM and NSC boundary conditions. We have performed this analysis for \( E = [10^{-8}, 10^{-4}] \), \( \Lambda = [10^{-3}, 1] \), \( Pm = [10^{-8}, 10^{-5}] \) and found a minimum for \( Ra(\sigma_i) \) at \( \sigma_i = 0 \) in all cases. This confirms that the stationary route discussed in Section III gives the actual scenario for the onset of convection with both NSM and NSC boundary conditions.

Interestingly, for \( \Lambda = 0 \), it is known that overstability may occur for \( Pr < 0.6766 \) (see Table X, page 119 of Ref. 14). The reason for this difference between the magnetic and non-magnetic case is most likely that in the limit \( E \to 0 \), the magnetic mode is always more unstable than the viscous mode and it is stationary. This holds in all cases we considered here (\( \Lambda > 10^{-3} \)), but overstability may occur for low enough values of \( \Lambda \), which are not relevant to the Earth’s core. Rotating magnetoconvection strongly differs from pure magnetoconvection too where the onset is strongly determined by Alfvén waves.26 With rotation, the most unstable mode is always stationary, so waves cannot play such a prominent role as in the pure magnetoconvective case. We found this to be true in the range of Elsasser number we considered, for which rotation tends to dominate. We anticipate that in regimes of significantly higher Elsasser numbers, electromagnetic effects would dominate and the mechanisms acting in pure magnetoconvection may come to the fore.

FIG. 13. Onset of the overstable (oscillatory) and stationary modes at \( Pm = 10^{-8} \) for different values of \( \Lambda \) and \( Pr \) for SFM boundary conditions. The critical Rayleigh number for the overstable mode is always higher than for the stationary one. This confirms that the stationary mode is the one responsible for the onset of convection in the range of parameters analysed.

FIG. 14. Normalised difference between the Rayleigh numbers for the marginally stationary cases and marginally overstable cases for NSM boundary conditions.
V. DISCUSSION

In this work, we have presented a detailed parametric study of the linear stability problem governing the onset of plane magnetoconvection down to asymptotic regimes in the limit $E \to 0$. This led to the following results:

1. We were able to precisely verify and quantify the theoretical scalings for the onset of the magnetic and the viscous convection modes, $Ra^m_c \propto \Lambda^{-1}$ and $Ra^v_c \propto E^{-1/3}$, for NSM, NSC, and SFM boundary conditions.

2. Our parametric analysis led us to establish a map in the space of parameters $(E, \Lambda)$ and to distinguish three regions: one where only the viscous mode exists, one where both viscous and magnetic modes exist but the magnetic mode is more unstable, and one where both exist but the viscous mode is more unstable. The crossover between instabilities due to the magnetic mode and instabilities due to the viscous mode occurs for $\Lambda = 7.22 E^{1/3}$ in the limit $E \to 0$, in agreement with Sreenivasan and Jones for SFM and NSM boundaries.

3. In the case of no-slip, electrically conducting boundaries (NSC), the cross-over occurs for $\Lambda = 3.6 E^{1/3}$, i.e., lower than when the boundaries are electrically insulating (NSM). In this sense, electrically conducting boundaries are more destabilizing.

4. The $\Lambda$–$E$ regime diagram for NSM conditions (Figure 11) explains why the viscous–magnetic transition was observed in the experiments of Nakagawa and not in the experiments of Aurnou and Olson.

5. The long-standing question of the occurrence of overstability has been clarified. We have shown that for a wide range of parameters ($E = [10^{-9}, 10^{-2}], \Lambda = [10^{-4}, 1], Pr = [10^{-5}, 1]$, and $Pm = [10^{-5}, 1]$), convection sets in through the instability of the stationary mode. This remains true even under the destabilising effect of electrically conducting boundaries. As such, rotating magnetoconvection at moderate Elsasser number occupies an intermediate ground between rotating convection at low $Pr$ ($<0.6766$) and pure magnetoconvection in the low diffusion limit, as in both cases, convection occurs through oscillatory modes (Alfvén waves in the second case).

Convection in planetary cores is one of the main motivations for the study of rotating magnetoconvection. Yet, it would be somewhat over-optimistic to expect ideal studies such as these to provide definite answers on the actual mechanisms underlying such a complex system. Nevertheless, it is tempting to try and extrapolate our results to planetary systems. Using values of $\Lambda$ between 0.08 and 1 and $E = 10^{-15}$, our results would suggest that the onset of the convection inside the Earth’s TC would be magnetically controlled. In the same way, our analysis can be applied to
Mercury, for which $\Lambda \sim 6 \times 10^{-5}$ \textsuperscript{27} and $E = 10^{-12}$.\textsuperscript{28} Then, asymptotic law (21) suggests that the convection in Mercury’s TC would set off in the viscous mode.

**ACKNOWLEDGMENTS**

The authors acknowledge financial support from the Leverhulme Trust, UK (Grant No. RPG-2012-456), and the Royal Academy of Engineering.