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Optimal magnet configurations for Lorentz force velocimetry in low conductivity fluids

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Abstract
We show that the performances of flowmeters based on the measurement of Lorentz force in duct flows can be sufficiently optimized to be applied to fluids of low electrical conductivity. The main technological challenge is to design a system with permanent magnets generating a strong enough field for the Lorentz force generated when a fluid of low conductivity passes through it to be reliably measured. To achieve this, we optimize the design of a magnet system based on Halbach arrays placed on either side of the duct. In the process, we show that the fluid flow can be approximated as a moving solid bar with practically no impact on the optimization result and devise a rather general iterative optimization procedure, which incurs drastically less computational cost than a direct procedure of equivalent precision. We show that both the Lorentz force and the efficiency of the system (defined as the ratio of the Lorentz force to the weight of the system) can be increased several fold by using Halbach arrays made of three, five, seven or nine magnets on either side of the duct but that this improvement comes at a cost in terms of the precision required to position the system.

Keywords: optimal magnet configurations, Lorentz force velocimetry, low conductivity fluids

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we shall numerically optimize magnet systems for the design of a Lorentz force flowmeter for fluids of low electrical conductivity. A vast number of industrial processes require the measurement of the flow rate of liquids. Achieving such measurements with a high degree of precision often brings major benefits in terms of reduction of manufacturing costs and product improvement [1, 2]. At the same time, modifying existing installations is often awkward and costly, so new measurement techniques must remain easy to implement and flexible as well as reliable. In this context, contactless methods potentially achieve an excellent compromise between these constraints. The idea of contactless methods based on the fluid’s electrical properties dates back to M Faraday’s first attempt to evaluate the stream of the Thames river, by measuring the voltage across it [3]. Nowadays, electromagnetic velocimeters are well developed and rely on two main principles: electric potential velocimetry (EPV) is the direct heir of Faraday’s idea. It consists of measuring the flow voltage induced by a moving fluid in an ambient magnetic field (the Earth’s magnetic field in Faraday’s example). This technique is widely used in the food industry, not only because of its proven efficiency in fluids of low electrical conductivity, but also for its high-precision measurements in laboratory experiments [4, 5]. Although non-intrusive, it does require direct contact with the fluid, which is sometimes difficult to achieve, depending on the electro-chemical properties of the fluid and the electrodes. Lorentz force velocimetry (LFV) is, by contrast, entirely contactless. Its principle was first proposed by Shercliff [6]. It consists of measuring either the force, the displacement or the velocity, which a conducting
fluid imprints on a magnet system when crossing the field it generates. An alternative, entirely contactless method was also recently proposed by [7], which relies on the measurement of the phase shift induced by the flow, between the ac magnetic field generated by an emitter-coil and the field captured by receiver-coils. LFV is very efficient in highly conducting metals and presents major advantages in corrosive, high-temperature melts, which would destroy any of the electrodes required for EPV [8, 9]. Its main shortcoming is that it requires high fluid conductivity to generate a force that can be reliably measured. Recently, however, the advent of high sensitivity force and displacement sensors made it possible to extend LFV to low conductivity fluids, such as electrolytes, which are typically 10^4 times less conductive than liquid metals [10]. In this case, the forces acting on the magnet system are of the order of 10^6 N, which is typically 10^8 to 10^9 times smaller than the magnet system weight. Since modern force sensors can detect forces above 10^5 N with good precision, it is crucial to maximize the Lorentz force generated to increase the measurement system efficiency and exceed this value, if one is to successfully apply LFV to low conductivity fluids. For this, the magnet system must be optimized.

In this paper, we take advantage of the strong magnetic fields generated by Halbach arrays [11], and develop a numerical method to optimize such a system. In view of applications, we restrict our analysis to small systems (magnet system weight must be less than 1 kg, and we seek to generate a force larger than 10^5 N). The main difficulty in the optimization procedure is that each point in the parameter space requires us to solve a complex set of PDE which governs both the electromagnetic problem and the equations of fluid motion. The originality of our solution lies in two points: first, we show that the flow can be replaced by a solid bar at practically no cost in terms of the precision of the results. Secondly, we devise an efficient numerical procedure, where the number of optimization points is reduced by successive grid refinement in the parameter space.

The paper consists of six sections. Section 2 describes the governing equations, problem geometry and physical assumptions. Section 3 is devoted to the description of the numerical system and its validation. Section 4 describes the optimization procedure and its validation. In section 5, the optimization results and their sensitivity are discussed. Finally, section 6 summarizes our findings.

2. Governing equations and physical assumptions

2.1. Problem geometry

The geometry of the considered problem is represented in figure 1 and corresponds to an experimental setup which is currently being developed at Ilmenau University of Technology [12]: it consists of a channel of rectangular cross-section of outer dimensions 0.054 m × 0.054 m, where an electrically conducting fluid (electric conductivity \( \sigma = 45 \text{ S m}^{-1} \)) flows along the \( x \) axis, with velocity \( v \), which averages over the cross-section to \( V_0 = 5 \text{ m s}^{-1} \). The channel wall thickness is \( w = 2 \text{ mm} \), so that the active flow cross-section is \( L_x \times L_z = 0.05 \text{ m} \times 0.05 \text{ m} \). The magnet system is located on either side of the channel and generates a magnetic field \( \mathbf{B} \) across the flow. Magnets are rectangular, of density \( \rho_m = 7500 \text{ kg m}^{-3} \) and fitted symmetrically, 1 mm away from each channel wall (not shown in figure 1), so the distance between the surfaces of the magnet poles is 0.056 m (see figure 1(a)). The origin is chosen on the channel centerline, at the center of the magnet system, with \( \mathbf{e}_y \) in the streamwise direction and \( \mathbf{e}_z \) along the main direction of the magnetic field in the channel. The remanence, magnetization and relative magnetic permeability of the magnets are respectively \( B_r = 1.09 \text{ T} \), \( M_0 = B_r/\mu_0 = 8.673 \times 10^3 \text{ A}\text{ m}^{-1} \) and \( \mu_r = 1 \). Fluid motion inside the magnetic field generates an electric current density \( \mathbf{J} \) in the fluid. By interacting with the primary magnetic field, this current generates a Lorentz force \( \mathbf{F}_L = \mathbf{J} \times \mathbf{B} \) which brakes the flow. By virtue of Newton’s third law (actio est reactio), the same force acts on the magnet system in the opposite direction, and Since it linearly depends on the flow rate, measuring this force provides an indirect measurement of the flow rate [8].

Several magnet arrangements are considered: a system of two magnets (\( N = 1 \) magnet on each side of the channel) and systems of two Halbach arrays (\( N = 3, 5, 7 \) and 9). Halbach arrays are magnet arrays with a predefined one-sided magnetization pattern, which produce high magnetic fields within their gap [11]. Figure 1(b) shows a Halbach array with \( N = 3 \) magnets on either side of the duct. These arrays
have the unique property that the whole magnetic flux escapes through one surface with none of it escaping through the opposite side. They also provide a nearly two-dimensional magnetization pattern. Halbach arrays are commonly used in various applications such as magnetic bearings, brushless ac motors, nuclear magnetic resonance (NMR) devices and magnetically levitating (maglev) systems [13].

2.2. Governing equations

The system of equations governing electromagnetic quantities is Maxwell’s equations for static quantities, in the approximation of a conducting medium, with the addition of Ohm’s law:

\[ \nabla \times \mathbf{H} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0, \]

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{B} = \mu (\mathbf{H} + \mathbf{M}). \]

Here, \( \mathbf{H} \) is the magnetic field, \( \mathbf{B} \) the magnetic flux density, \( \mathbf{E} \) the electric field, \( \mu \) the absolute permeability and \( \mathbf{M} \) the magnetization of the magnets. We shall introduce the magnetic vector potential \( \mathbf{A} \) and the electric potential \( \phi \):

\[ \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi, \]

which are uniquely defined by the addition of Gauss’ gauge

\[ \nabla \cdot \mathbf{A} = 0, \]

and by imposing \( \phi = 0 \) at infinity (in subsequent numerical computations, this shall be applied at the boundary of the domain, far from the magnets). With the help of these new variables, the original system of equations can be reduced to simpler equations governing the evolution of \( \mathbf{A} \) and \( \phi \). Furthermore, we shall use non-dimensional variables, defined for each quantity \( G \) as \( \tilde{G} = G/G_{\text{ref}} \), where dimensional reference values \( G_{\text{ref}} \) for lengths, velocities, magnetization, magnetic flux density, magnetic vector potential, electric potential, electric current density and Lorentz force are respectively \( L = L_0, \; V_0, \; M_0, \; \mu_0 M_0, \; \mu_0 M_0 L, \; \mu_0 V_0, \; \sigma \mu_0 V_0 M_0, \) and \( \sigma \mu_0 V_0 M_0 \). Finally, the non-dimensional system of equations governing this problem is written:

\[ \tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\mathbf{A}} - \tilde{\mathbf{M}}) = \tilde{R}_m (\tilde{\nabla} \times \tilde{\nabla} \tilde{\mathbf{A}} - \tilde{\nabla} \tilde{\phi}), \]

\[ \tilde{\nabla}^2 \tilde{\phi} = \tilde{\nabla} \cdot (\tilde{\nabla} \times \tilde{\mathbf{A}}). \]

The system turns out to be governed by a unique non-dimensional number, the magnetic Reynolds number \( \tilde{R}_m = \frac{\sigma \mu_0 V_0 L}{\sigma \mu_0 V_0 L} \), which is extremely small in this problem (\( \tilde{R}_m \approx 1.3 \times 10^{-6} \ll 1 \)). The right-hand side of equation (10) can therefore be neglected, in the frame of the low-\( \tilde{R}_m \) approximation (see [15] for its full derivation). Physically, this equates to assuming that currents are generated by the interaction between the flow and the magnetic field but that although these currents are sufficiently strong to generate the Lorentz force, the magnetic field they induce (through Ampere’s law (1)) is very small compared to the externally imposed one (here, the field generated by the permanent magnets). Consequently, the external magnetic field can be considered as imposed on the flow and \( \tilde{\mathbf{A}} \) is solely determined by electromagnetic quantities and the geometry. The electric potential \( \tilde{\phi} \), on the other hand, is determined by the current generated in the flow and still depends on \( \tilde{\mathbf{v}} \). This approximation, also called quasi-static, has been tested in a number of configurations, such as duct flows and turbulent flows [16, 17], and was found to lead to results that are practically indistinguishable from solutions of the full equations as soon as \( \tilde{R}_m < 0.1 \). The main benefit of the low-\( \tilde{R}_m \) approximation is that \( \tilde{\mathbf{A}} \) and \( \tilde{\phi} \) are mathematically decoupled, and can be calculated by first solving a simpler version of (10):

\[ \tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\mathbf{A}} - \tilde{\mathbf{M}}) = 0, \]

and then solving (11) using the solution of (12) for \( \tilde{\mathbf{A}} \). Finally, the intensity of the force felt by the magnet system, which we shall be aiming at maximizing, is that of the Lorentz force felt by the fluid, integrated over the whole fluid domain \( D_f \):

\[ \tilde{\mathbf{F}}_L = \int_{D_f} \tilde{\mathbf{J}} \times \tilde{\mathbf{B}} \, dx \, dy \, dz. \]

Tests of the low-\( \tilde{R}_m \) approximation for the geometric configuration of interest in this paper were performed in [18]. It was found to incur a relative error of \( 2 \times 10^{-5} \) on the value of the Lorentz force in the two-magnet configuration of figure 1(a). For the rest of the paper, quantities will be non-dimensional, unless otherwise stated, and the tilde will be omitted.

2.3. Boundary and symmetry conditions

At the surface of the magnets and the electrolyte, continuity boundary conditions are imposed for the magnetic field:

\[ \mathbf{n} \times (\nabla \times \mathbf{A}_1 - \nabla \times \mathbf{A}_2) = 0, \]

where \( \mathbf{n} \) is the outward normal to the surface, and subscripts 1 and 2 respectively refer to the domains occupied by the magnets and by air. At the boundaries of the computational domain, far from the magnet, a magnetic insulation boundary condition is applied:

\[ \mathbf{n} \times \mathbf{A} = 0. \]

The conditions for the electric potential are those of electric insulation at the lateral boundaries of the fluid domain:

\[ \mathbf{n} \cdot \nabla \phi = 0, \]

and perfectly conducting boundary at the upstream and downstream ends of the duct:

\[ \phi = 0. \]

Since the inlet and outlet of the duct are two artificial boundaries, we apply artificial boundary conditions. These conditions are acceptable if they have the smallest effect on the solution. The common approach for scalar fields is to use a soft condition (16). This is discussed in textbooks on CFD, for example in Zikanov’s book [31].
Nevertheless, most of the electric current is generated in the vicinity of the magnets and very little is present near the end of the duct, so using one or the other condition is not expected to influence the result much. This was verified by performing two numerical simulations in the conditions described in section 3.3, for N = 1. The local difference in the distribution of electric current was indeed below numerical precision (0.01%). Outside the fluid domain, all materials are insulating so there is no electric current. For the system with two permanent magnets magnetized along the y axis (see figure 1(a)), only one quarter of the full geometry can be considered. Here, the boundary conditions (15), (17) were applied on the symmetry plane z = 0 and additional boundary conditions on the symmetry plane y = 0 were applied as follows:
\[ \mathbf{n} \times (\nabla \times \mathbf{A}) = 0, \quad (18) \]
\[ \mathbf{n} \cdot \nabla \phi = 0. \quad (19) \]
In the case of the magnet system with two Halbach arrays only symmetry with respect to the z = 0 symmetry plane can be used with boundary conditions (15), (17), because the magnetization directions inside the magnets are non-symmetric with respect to x = 0 and y = 0 symmetry planes (see figure 1(b)).

2.4. The solid bar approximation

Until now, only the electromagnetic part of the problem has been considered. To close the system of equations, the Navier–Stokes equations which govern the evolution of velocity and pressure distributions in the duct should be added. In practical applications where the flow rate is high, however, the flow in the duct is turbulent, with intense fluctuations, which, fortunately, we need not face for the purpose of this work. As mentioned in section 2.2, in the low-Rm approximation, the magnetic flux \( \mathbf{B} \) is independent of the velocity field \( \mathbf{v} \), so the Lorentz force (13) only depends on \( \mathbf{v} \) through the electric current \( \mathbf{J} \). From (11), this dependence is linear, and so is therefore that of \( \mathbf{F}_L \) on \( \mathbf{v} \). This implies that the fluctuations of \( \mathbf{v} \) do not contribute to the time average of \( \mathbf{F}_L \), and that only the time average of \( \mathbf{v} \) does. Since the typical mean velocity profile of a turbulent duct flow is essentially flat in the bulk of the flow, with boundary layers of thickness \( \delta \) developing along each of the duct walls, the local error in \( \mathbf{v} \) incurred by assuming that the flow behaves like a solid, moving at velocity \( \mathbf{v} = V_0 \mathbf{e}_z \), and, in turn on \( \mathbf{F}_L \), is of order \( \delta/L \). We shall see in section 3 that the (global) error on \( \mathbf{F}_L \) incurred by replacing the turbulent flow profile with a constant velocity profile across the duct is significantly smaller than the expected precision of the measurement system. The solid bar approximation was also successfully used in previous studies on LFV based on time of flight measurements [19] and on LFV in an open channel geometry [20].

3. Numerical system and validation

3.1. Description of the numerical system

The ac/dc module of the commercial software package COMSOL Multiphysics was used to solve the system of equations (11) and (12) subject to boundary conditions (14)–(19) [21]. COMSOL is a standard implementation of the finite elements method, with, in the present case, elements of the second order. An iterative solver based on the FGMRES method (flexible generalized minimum residual method) was used to solve the linear system (22–24), with relative tolerance set to \( 10^{-6} \). Geometric multigrid was used as a preconditioner of the linear system solver [25]. The successive over-relaxation methods SOR and SORU were respectively used as presmoother and postsmoother for the preconditioner of the linear system solver [23]. The iterative GMRES solver was used as the coarse solver for the preconditioner of the linear system solver [26, 22]. Finally, the SSOR (symmetric successive over-relaxation) method was used as the preconditioner of the coarse solver [23].

3.2. Mesh description and convergence test

We first estimate mesh precision required for an accurate calculation of electromagnetic quantities by comparing numerical computations of the magnetic field in a configuration where the magnetic field can be easily calculated analytically. For the sake of concision, we shall restrict the test to the case of the two-magnet system depicted in figure 1(a).

The meshes used for larger magnet systems rely on the same principles and achieve the same resolution. Three different mesh sizes were used inside the electrolyte and magnets to obtain the y-component of the magnetic field along the y axis at \( x = z = 0 \), along the z axis at \( x = y = 0 \) and along the x axis at \( y = z = 0 \). The magnets had fixed dimensions: \( l_z = l_y = 1 \) (along the x and z axes) and \( l_x = 0.5 \) (along the y axis).

The air domain dimensions were 14 along the x axis and 10 along the y and z axes. The symmetry conditions were used on the \( x = 0, y = 0 \) and \( z = 0 \) symmetry planes to reduce the size of computational domain by a factor 8. All sub-domains were discretized using second-order Lagrangian tetrahedral vector elements. Convergence under mesh refinement shall be tested by comparing numerical results to the analytical solution found by the author of [27], adapted to our configuration. Denoting the positions of the magnet edges by \( x_m, y_m \), and \( z_i \), and assuming that the magnet is magnetized along the y axis, the y-component of the magnetic flux density at a point with coordinates \( x, y \) and \( z \) outside the magnet is given by
\[
B_y(x, y, z) = \frac{\mu_0 \cdot M}{4\pi} \sum_{k=1}^{2} \sum_{n=1}^{2} (-1)^{k+n+m} \frac{\arctan \left( \frac{(y-y_m)(z-z_k)}{(x-x_m)^2+(y-y_m)^2+(z-z_k)^2} \right)}{(x-x_m)(z-z_k)} , \quad (20)
\]

The principle of superposition was adopted to find the total magnetic field component \( B_y \) in any point between two magnets described above, which were equally magnetized along the y axis. The relative error between the analytically and numerically obtained magnetic field component \( B_y \) was estimated as
\[
\epsilon = \sqrt{\frac{\sum_i (B_{y,i} - B_{y,ij})^2}{\sum_i (B_{y,i})^2}}, \quad (21)
\]
where \(B_{yi}\) and \(B_{zi}\) respectively denote the value of analytically and numerically obtained values of the \(y\)-field component at the \(i\)th point of the mesh. Table 1 summarizes the features of each mesh as well as the result of the convergence test. Figure 2 shows the profiles of \(B_{i}(0, y, 0)\) and \(B_{i}(x, 0, 0)\) obtained analytically and numerically. Profile \(B_{i}(0, 0, z)\) is not shown as it is similar to profile \(B_{i}(x, 0, 0)\). The relative error is very small in all cases and decreases when decreasing the element size inside the magnets and electrolyte from \(\Delta x^M = 0.12\) to \(\Delta x^M = 0.08\), but does not from \(\Delta x^m = 0.08\) to \(\Delta x^m = 0.04\), as the software’s internal numerical precision was reached. For the rest of the paper, a maximum element size \(\Delta x^M \leq 0.08\) (Mesh M2) shall be used inside the electrolyte and the magnets, because it is optimal both in terms of accuracy and computational cost.

### 3.3. Validation of the solid bar approximation

We shall now test the numerical system and the physical approximations it relies on by comparing it to the experimental results of [10]. The reference geometry is similar to that shown in figure 1(a) but the experimental parameters slightly differ from data given in subsection 2.1, however. The active channel cross-section was 0.026 m \(\times\) 0.046 m, with a 2 mm wall thickness so that the corresponding non-dimensional geometric parameters were \(L_z = 1.769\) and \(w = 0.077\). The maximum average velocity of the electrolyte was fixed to \(V_0 = 4\) m s\(^{-1}\). The magnet system contained two magnets of dimension \(l_x = l_y = 1.154\) and \(l_z = 2.692\), and distance 1.23. Their magnetic properties \((B_r = 1.34\) T and \(\mu_r = 1)\) were not given in [10] so they were calculated so as to recover the magnetic field component \(B_y\) measured experimentally at the center of the magnet system.

We performed three numerical simulations. Table 2 lists the most important parameters of the corresponding numerical models as well as the full Lorentz force obtained in each model. The three numerical models differ from each other both through the physical assumptions they rely on and through their meshes: models 1 and 2 shall assume a solid body motion, with constant velocity of the electrolyte, while model 3 shall feature turbulent flow of the electrolyte, which was calculated separately using a \(k–\epsilon\) turbulence model and a logarithmic wall function, fitted to experimental data. The detail of this separate numerical work is reported in [18]. Model 1 features a less refined mesh than models 2 and 3. Also, in order to save resources, the electrolyte domain and the air domain were shortened to \(L_x = 19.2\) in the streamwise direction, since the

### Table 1. Mesh parameters and mesh convergence test. \(\Delta x^M\) and \(\Delta x^m\) denote respectively the maximum and minimum element size inside the electrolyte and magnets. \(\Delta x^M\) and \(\Delta x^m\) denote the maximum and minimum element size inside the air domain. All element sizes are normalized by \(L_x = 0.05\) m.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M(_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta x^M)</td>
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<td>0.06</td>
<td>0.06</td>
<td>0.08</td>
</tr>
<tr>
<td>(\Delta x^m)</td>
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<td>0.02</td>
<td>0.04</td>
<td>0.08</td>
</tr>
<tr>
<td>Number of elements</td>
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<td>73742</td>
<td>461103</td>
<td>137903</td>
</tr>
<tr>
<td>Number of degrees of freedom (DoF)</td>
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<td>585269</td>
<td>3595638</td>
<td>1089268</td>
</tr>
<tr>
<td>CPU time (s)</td>
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<td>17.3</td>
<td>121.5</td>
<td></td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>0.0039</td>
<td>0.0024</td>
<td>0.0028</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2. Comparison of numerical models: solid body versus turbulent flow. RANS and MEF stand for Reynolds averaged Navier–Stokes and magnetic and electric field equations, respectively.

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of motion</td>
<td>solid body</td>
<td>solid body</td>
<td>turbulent flow</td>
</tr>
<tr>
<td>Cells in boundary layer</td>
<td>–</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>(\Delta x^M)</td>
<td>0.096</td>
<td>0.096</td>
<td>0.096</td>
</tr>
<tr>
<td>Total number of elements</td>
<td>255 326</td>
<td>2227 992</td>
<td>2227 992</td>
</tr>
<tr>
<td>Total number of DoF</td>
<td>1997 880</td>
<td>17 265 894</td>
<td>17 265 894</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>80</td>
<td>9173</td>
<td>6236 (RANS)+11707(MEF)</td>
</tr>
<tr>
<td>Full (F_i)</td>
<td>(4.997 \times 10^{-2})</td>
<td>(4.999 \times 10^{-2})</td>
<td>(4.957 \times 10^{-2})</td>
</tr>
<tr>
<td>(\epsilon_F)</td>
<td>0.3002</td>
<td>0.3013</td>
<td>0.2871</td>
</tr>
</tbody>
</table>
velocity profile is constant in this model and does not change along the flow direction. In models 2 and 3, on the other hand, the full length of the electrolyte domain and air domain was considered ($L_z = 57.7$).

Figure 4 plots the magnetic flux density $B_y$, velocity $V_x$, eddy current density $J_z$ and Lorentz force density $f_{Lx}$ against $y$ axis at $x = 0$ and $z = 0$ obtained using models 2 and 3. The results from model 1 are almost the same as in model 2 and therefore not shown here. It can be seen that different velocity profiles in the channel induce different eddy currents and Lorentz force distributions in the fluid. In the vicinity of the wall, the solid body velocity is higher than the velocity of the turbulent flow (see figure 4(b)). The current density and the Lorentz force density are also correspondingly higher (see figures 4(c) and (d)). Conversely, since the same flow rate was imposed in all models, the fluid velocity was higher in the bulk of the turbulent flow than for the solid bar and the Lorentz force density there is correspondingly higher. Overall, since the magnetic field is mostly transverse and varies little across the channel, the total Lorentz force mostly depends on the flow rate and the departure between its values for the turbulent flow and the solid bar is minimal. Since it is clear from table 2 that the obtained Lorentz drag force is almost the same for all numerical models and since the CPU time of model 1 is much less than that of models 2 and 3, the features of model 1 will be used for the rest of the paper. The corresponding mesh is represented in figure 3.

The values of the Lorentz force obtained numerically with all three numerical models and those obtained experimentally by the authors of [10] are plotted against the velocity of the electrolyte in figure 5: the difference between numerics and experiment is well below experimental precision. Consequently, this confirms that considerable computational cost is spared at hardly any cost in precision by using model 1. We shall thus use the features of model 1 in all subsequent calculations.
On the top of the validation of the model against experimental data, we shall estimate the sensitivity of the method to the velocity profile by comparing the solid bar approximation to the analytical average turbulent velocity profile given in [30]. The analytical expression for the circular section of the duct as follows:

\[
V = \frac{\alpha}{(1 + \alpha) \ln(1 + \alpha) - \alpha} \ln(1 + \alpha(1 - 4\alpha^2)(1 - 4\alpha^2)),
\]

(22)

where \(\alpha\) is the parameter defining the flow regime: \(\alpha \to 0\) corresponds to Poiseuille-shaped laminar profile, whereas \(\alpha \to \infty\) corresponds to turbulent flow profile. Clearly, this ad hoc profile does not provide a precise representation of a true turbulent duct flow, but does qualitatively reproduce the alteration in the profile shape which occurs when the flow span regimes from laminar to turbulent. The problem data are those from section 2.1. The magnet system under consideration contained two magnets \((N = 1)\), as shown in figure 1(a). The magnet dimensions were fixed to \(l_x = l_y = 1\) and \(l_z = 0.5\), and the length of the electrolyte and air domains along the \(x\) axis to \(L_x = 10\). One quarter of the problem was considered taking the advantage of the symmetry conditions on the \(y = 0\) and \(z = 0\) symmetry planes. Features of the mesh are reported in table 1 (Mesh \(M_0\)). The solution was obtained using the methods described in section 3.

Figure 6(a) depicts the velocity profiles at different flow regimes for different values of parameter \(\alpha\). At higher \(\alpha\), flow becomes more turbulent. The Lorentz force approaches its highest value as \(\alpha \to \infty\) (see figure 6(b)), and the difference between \(F_L\) for solid body approximation and turbulent profile vanishes in this limit. A similar result was found by [30] for the pipe flow. Nevertheless, a more realistic velocity profile corresponds to not very large \(\alpha\), at which the difference of force in comparison to the solid body case is about 5% (the case of \(\alpha = 10^5\)). Moreover, the effect of the velocity profile on the Lorentz force and on the optimizer was investigated in work [18]. The results showed that the Lorentz force and optimizer obtained for solid body approximation are affected insignificantly (less than 2%) when the flow is highly turbulent.

4. Optimization

4.1. Optimization procedure

With a tool enabling us to numerically calculate the force felt by the magnet system \(F_L\), we shall now optimize the magnet arrangement in order to maximize \(F_L\). The magnet system is made of two Halbach arrays [14] placed opposite to each other on either side of the channel. Each array consists of \(N\) rectangular magnets, with magnetization pointing alternately in the \(x\) and \(y\) directions (see figure 7). The two arrays are designed so that opposite pairs of magnets are of the same dimension. All magnets have the same size \(l_x\) and \(l_y\) in the \(y\) and \(z\) directions but magnets magnetized upstream and downstream may have different dimensions \(l_{xz}\) from magnets magnetized spanwise (see figure 7(b)). The optimization is subject to the constraint of keeping the magnet system below a given mass \(m\), which in the present case shall be 1 kg.

Mathematically, finding the magnet system, parameterized by variables \((l_x, l_{xz}, l_y, l_z)\), that maximizes the Lorentz force is formulated as the optimization problem of maximizing \(F_L\), subject to the following constraints:

\[
W(l_x, l_{xz}, l_y, l_z) \leq W^M
\]

(23)

\[
l_x^M \leq l_x \leq l_x^M
\]

(24)

\[
l_y^M \leq l_y \leq l_y^M
\]

(25)

\[
\]
The optimization problem (23)–(28), where $F_L$ at one point involves the numerical resolution of a PDE, the computational cost would become prohibitive even at moderate resolutions. We shall therefore adopt an alternative approach, which is much less computationally intensive, and incurs practically no penalty on the final precision attained. The algorithm can be summarized as follows (see also the sketch in figure 8): taking advantage of the smooth variations of $F_L$ against each of the design variables, we first calculate $F_L$ on a coarse grid, with $n_v$ values per design variable, thus spaced by $\Delta l = (l^M - l^m)/(n_v - 1)$, (step 1). These ‘exact’ values are fitted with a multivariate polynomial of order $q$ (step 2), which is used in place of $F_L$ to solve the optimization problem (step 3). If the desired precision on the location of the maximum is achieved, then the procedure ends, otherwise the grid is refined with a new grid, with each variable spanning a half-size interval centered on the location of the coarse maximum, thus with $\Delta l^{\text{new}} = \Delta l/2$. The procedure is then iterated until the desired precision

$\epsilon_l = \left| l^{\text{new}}_{i, \text{opt}} - l^{\text{old}}_{i, \text{opt}} \right| = \frac{1}{n_{DV}} \sum_{i=1}^{n_{DV}} \sqrt{(l^{\text{new}}_{i, \text{opt}} - l^{\text{old}}_{i, \text{opt}})^2} \leq 10^{-3}$ (29)

is reached. Here $n_{DV}$ is the number of design variables and $l^{\text{old}}_{i, \text{opt}}$ and $l^{\text{new}}_{i, \text{opt}}$ are the values taken by the optimizer at successive iterations.

The optimization problem (23)–(28), where $F_L$ is replaced by a polynomial of order $q$ is solved by the standard Lagrange multipliers method. In practice, the associated Karush–Kuhn–Tucker equations were solved numerically to a relative optimization tolerance of $10^{-9}$ by means of the $fmincon$ MATLAB function, in which this method is implemented [28, 29].

4.2. Validation of the optimization procedure

The optimization procedure shall be validated by comparing optimal magnet systems obtained with two different meshes of the parameter space: one with $\Delta l = 0.1$ ($n_v = 3$) and a finer one with $\Delta l = 0.05$ ($n_v = 5$). The values of $F_L$ on these two grids shall respectively be fitted with second- and fourth-order polynomials. The test is performed for $N = 1$ with $W^M = 0.3413$ (two-magnet system) and $N = 3$ with $W^M = 1.067$ (two-Halbach array system). The results are gathered in table 3.
that the error on \( \Delta l \) when at the same time the computation cost increases five fold, which shall respectively span the intervals \( \Delta l \). Furthermore, the relative RMS average of the error on \( F_L \) evaluated at the points of the coarser mesh is 0.25% (\( N = 1 \)) and 0.24% (\( N = 3 \)), thus confirming that the error on \( F_L \) remains locally low too. Consequently, we shall use \( \Delta l = 0.1 \) and \( q = 2 \) for the rest of the paper.

5. Optimization results

5.1. Optimal magnet systems based on Halbach arrays

We shall now apply the optimization procedure validated in the previous section to find the magnet arrangement that yields the highest Lorentz force for a given maximum magnet system mass \( W^M \) and for a given number of magnet pairs \( N \), which shall respectively span the intervals \( W^M \in [0 \ 1.1] \) and \( N \in \{1, 3, 5, 7, 9\} \). The initial optimization parameters for all cases studied are gathered in table 4.

![Table 3](image)

**Table 3.** Precision test on the optimization procedure with polynomials. The relative precision is calculated in reference to the polynomial of highest order.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \Delta l )</th>
<th>( n_c )</th>
<th>( q )</th>
<th>( F^\text{opt} )</th>
<th>( W^\text{opt} )</th>
<th>( F^\text{opt}/W^\text{opt} )</th>
<th>( F^\text{opt}/W^\text{opt} )</th>
<th>( F^\text{opt}/W^\text{opt} )</th>
<th>( F^\text{opt}/W^\text{opt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>3</td>
<td>2</td>
<td>3.397 \times 10^{-3}</td>
<td>0.3413</td>
<td>0.536</td>
<td>0.354</td>
<td>0.898</td>
<td>n/a</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>5</td>
<td>4</td>
<td>3.404 \times 10^{-3}</td>
<td>0.3413</td>
<td>0.540</td>
<td>0.350</td>
<td>0.904</td>
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</tr>
<tr>
<td>Relative error</td>
<td>0.0021</td>
<td>0</td>
<td>0</td>
<td>0.0074</td>
<td>0.0114</td>
<td>0.0066</td>
<td>n/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>3</td>
<td>2</td>
<td>1.844 \times 10^{-2}</td>
<td>1.067</td>
<td>0.51</td>
<td>0.396</td>
<td>0.980</td>
<td>0.350</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>5</td>
<td>4</td>
<td>1.848 \times 10^{-2}</td>
<td>1.067</td>
<td>0.51</td>
<td>0.394</td>
<td>0.988</td>
<td>0.346</td>
</tr>
<tr>
<td>Relative error</td>
<td>0.0022</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0051</td>
<td>0.0081</td>
<td>0.0116</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In both cases, the increase in precision gained by refining the mesh on the optimal dimensions and weight is below 0.1%, when at the same time the computation cost increases five fold (\( N = 1 \)) and eightfold (\( N = 3 \)). Furthermore, the relative RMS average of the error on \( F_L \) evaluated at the points of the coarser mesh is 0.25% (\( N = 1 \)) and 0.24% (\( N = 3 \)), thus confirming that the error on \( F_L \) remains locally low too. Consequently, we shall use \( \Delta l = 0.1 \) and \( q = 2 \) for the rest of the paper.

5. Optimization results

5.1. Optimal magnet systems based on Halbach arrays

The maximum streamwise component of the Lorentz force \( F_L^\text{opt} \) and its ratio to the maximum system weight \( W^M \) are plotted against \( W^M \) in figures 9(a) and (b), respectively. The ratio of the dimensional Lorentz force felt by the magnet system to its weight \( EF = F_L^\text{opt}/(mg) \) gives a measure of the efficiency of the system and should therefore be maximized. Unsurprisingly, the maximum Lorentz force increases with the maximum allowed mass of the magnet system, regardless of the chosen magnet configuration (parameterized by \( N \)) and the optimal mass is always the maximum mass imposed \( (W^\text{opt} = W^M) \). For a given mass, however, the Lorentz force increases with \( N \) but saturates. Interestingly, the variations of \( F_L^\text{opt} \) are strongly sublinear for smaller \( N \) but become more and more linear for larger \( N \). Thus, for low mass systems \( (W^M < 0.4) \), \( N = 3 \) already nearly produces the maximum Lorentz force, whereas for \( W^M \in [0.4 \ 1] \), \( N = 7 \) is the best compromise between the optimal Lorentz force and simplicity, as the Lorentz force is practically the same as for \( N = 9 \) over this interval. One can expect that more complex magnet systems become advantageous for \( W^M > 1 \). The improved linearity at higher \( N \) can be understood as follows: for \( N = 1 \), the optimal magnet height \( H^\text{opt} \) is higher than the height of the electrolyte duct when \( m/m_0 \in [0.74 1 \ 0.67] \). This incurs a leakage of the magnetic flux in the air gap between the magnets and the electrolyte and a corresponding loss of the Lorentz force. Since \( H^\text{opt} \) varies with \( W^M \), so does its relative importance. Therefore, the dependence \( F_L^\text{opt}(W^M) \) is nonlinear in this case. When \( N > 1 \), \( H^\text{opt} \) is always smaller than the duct height so there is no leakage of magnetic flux in this region, and \( F_L^\text{opt}(W^M) \) becomes close to linear as soon as \( N > 1 \). Figure 9(b) shows that replacing a two-magnet system with
any Halbach array typically doubles or triples the efficiency of the system. Nevertheless, the variations of the ratio $E_F$ versus $W^M$ are more complex than those of $F_{\text{opt}}^L$, while it decreases with $W^M$ for $N = 1$ but presents a maximum for $N > 1$, this maximum is not very sharp and occurs at mildly increasing magnet’s weight when $N$ increases. It also varies little in value between $N = 3$ and $N = 9$. While $N = 3$ brings a significant enhancement of this ratio over $N = 1$, more complex magnet systems ($N > 3$) bring relatively little further improvement.

Optimal dimensions and aspect ratios of the magnets for $N = 1, 3, 7$ and $9$ are plotted against the maximum magnet system’s weight in figures 10 and 11. All optimal dimensions increase with $W^M$. For $N = 1$, they increase sublinearly with $W^M$ and the optimal streamwise aspect ratio of the magnets $l_{x,\text{opt}}/l_{x,\text{opt}}^0$ decreases from 1.68 (at $W^M = 0.3413$) to 1.36 (at $W^M = 1.067$), as shown in figure 11(a). The optimal spanwise aspect ratio of the magnets $l_{y,\text{opt}}/l_{y,\text{opt}}^0$ increases from 0.66 (at $W^M = 0.3413$) to 0.76 (at $W^M = 1.067$). Thus, heavier

Figure 10. Optimal dimensions of the magnets for: (a) $N = 1$, (b) $N = 3$, (c) $N = 7$ and (d) $N = 9$ against the maximum magnet system’s weight.

Figure 11. Optimal aspect ratios of the magnets for: (a) $N = 1$, (b) $N = 3$, (c) $N = 7$ and (d) $N = 9$ against the maximum magnet system’s weight.
optimal systems made of a single pair of magnets are closer to a cubic shape. A similar tendency is observed for magnet systems based on Halbach arrays ($N > 1$). The dependence of the optimal aspect ratios on $W_M$ however becomes weaker as $N$ increases, and the optimal magnet shape tends to become shorter along the streamwise and spanwise direction: for $N = 9$, the optimal ratio $l_{opt} / l_{W_M}$ is almost constant (around 2.43) while the ratio $d / l_{opt}$ varies from 0.53 (at $W_M = 0.427$) to 0.67 (at $W_M = 1.067$). Also, odd magnets (magnetized along $e_z$) are slightly more elongated in the $x$ direction than the even magnets (magnetized along $e_y$). This tendency is accentuated for heavier systems with a ratio $l_{opt} / d_{opt}$ decreasing from 0.98 (at $W_M = 0.427$) to 0.88 (at $W_M = 1.067$), as shown in figure 11(d).

Finally, the features of the magnet systems yielding the highest efficiency $E_F$ for each value of $N$ are gathered in table 5. Overall, in the interval of $W_M$ we considered, the most efficient magnet system is a Halbach array with $N = 9$, since it provides the highest Lorentz drag force and efficiency. One could expect that the performances of the magnet system would continue to increase at higher values of $N$. The corresponding optimal mass would, however, increase beyond the values of interest in this paper, which were set to ensure a relatively compact system.

Finally, it should be noted that an alternative approach to optimizing $F_L$, could have been to optimize the efficiency $E_F$. The methods described in this paper can certainly be easily adapted for this purpose. To demonstrate the applicability of LFV to electrolyte flows, however, it was first necessary to show that a force $F_L$ could be obtained that was at all detectable by commercially available sensors so it was crucial to optimize $F_L$ itself. Nevertheless, since we performed this analysis for a number of different values of $W_M$, we found that this optimal value also corresponded to the maximum of $W_M$, and then extracted the case that yielded the maximum of $F_L^{opt} / W_M$, the final result also optimizes the ratio $F_L / W_M$, albeit on a discrete set of values of $W_M$. Obviously, the result is a little less precise than if $E_F$ had been optimized directly, but allowed us to discuss both optimizations of $E_F$ and $F_L$.

5.2. Sensitivity analysis

To complete our analysis, we shall now evaluate how robust some of the optimal designs found in the previous section are to controlled or uncontrolled variations of the parameters defining the problem. The sensitivity for the magnet system with $N = 1, 5$ and $9$ was analyzed. The relative sensitivity of the maximum Lorentz force to any variable $X$ is defined and estimated as

$$S_X (F_L) = \frac{\partial F_L}{\partial X} (X^{opt}) \approx \frac{F_L^{opt} (1 + \eta) X^{opt} - F_L^{opt} (X^{opt})}{\eta F_L^{opt}}.$$  

(30)

We shall arbitrarily evaluate $S_X$ for a relative increase of $\eta$ in $X$, $X$ being in turn, $V_0, \sigma_0, M_0, y_0$ and $z_0$, where $y_0$ and $z_0$ were the $y$- and $z$-positions of the duct centerline (we have assumed $y_0 = z_0 = 0$ until now). The relative increase $\eta$ was chosen to be 5% for all disturbed parameters, except for $y_0$ for which $\eta = 1\%$, because $y_0$ must remain below the value of the air gap between the magnets and the duct $w_g = 0.02$. The sensitivity of design variables $l_x, l_z, l_y$ and $l_z$ is defined in a similar way.

Tables 6, 7 and 8 summarize the results of sensitivity analysis. It is noteworthy that the optimizer is practically insensitive to $V_0, \sigma_0$ and $M_0$ for all $N$. In practice, this means that once optimized, an LFV system can be used in the same

<table>
<thead>
<tr>
<th>$N$</th>
<th>$l_x^{opt}$</th>
<th>$l_y^{opt}$</th>
<th>$l_z^{opt}$</th>
<th>$l_w^{opt}$</th>
<th>$F_L^{opt}$</th>
<th>$F_L^{opt(d)} (N)$</th>
<th>$W^{opt}$</th>
<th>$E_F$</th>
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<tr>
<td>1</td>
<td>0.536</td>
<td>0.354</td>
<td>0.898</td>
<td>–</td>
<td>3.397 x 10^{-3}</td>
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geometry for different fluids and flow rates. Secondly, unlike $V_0$, $\sigma_0$ and $M_0$, the relative position of the duct with respect to the magnet system ($y_0, z_0$) influences both the optimizer and the resulting Lorentz force. In particular, $S_{yy}(F_L)$ is positive and increases with $N$ and $S_{zz}(F_L)$ is negative and decreases with $N$. It is therefore crucial to ensure a perfectly symmetric position of the magnet system with respect to the duct. The degradation in performance is relatively limited but for more complex systems ($N > 3$), an error in positioning of a few per cent could impact the effective precision of the flow rate measurement.

The fact that $S_{yy}(F_L) > 0$ seems to indicate that centering the magnets along the $y$ axis is not optimal. In practice however, it is due to the fact that since the magnetic field decreases strongly nonlinearly away from the magnet surface, the extra force gained by moving the magnets closer to the duct on one side is much larger than the force lost by moving the magnets away from the duct by the same distance on the other side. $S_{yy}(F_L) > 0$ therefore simply reflects that magnets should be placed as closely as possible to the duct on both sides. In practice, the distance to the wall always results from compromises imposed by the environment (such as accessibility or thermal constraints etc). The fact that this value is strongly positive points out that the penalty in performance incurred by not minimizing this distance is stronger than on any other parameter.

6. Conclusion

A computationally efficient optimization method was designed to determine the optimal dimensions of magnet systems in Lorentz force velocimeters. The originality of the method lies in two points. First, the mathematical model governing the system was simplified, thanks to two main physical approximations: the low-$Rm$ approximation allowed us to decouple electromagnetic and mechanical variables, and approximating the flow in the duct by a shortened solid bar both eliminated mechanical variables and reduced the computational effort. A series of numerical tests showed that the impact of these approximations remained orders of magnitude below the expected precision of such a velocimeter, thus establishing their validity. Second, the cost of the mathematical optimization procedure itself was drastically reduced by iteratively refining an initially coarse grid around the optimal point in the parameter space. The validation procedure can easily be reproduced to extend this combined technique to a much wider range of magnet shapes and geometries and to problems beyond the optimization of Lorentz force velocimeters, which originally motivated our approach.

Once validated, the new technique was applied to the design of a compact Lorentz force velocimeter with magnets arranged in a Halbach array, a design that involved up to four optimization parameters. The optimization problem was subject to linear constraints (the system dimensions were limited) and nonlinear constraints (the weight of the magnet system had to be less than 1 kg) to keep a compact system. It was found that Halbach arrays provide a significant improvement of performances over single magnet systems: for a given weight, systems with a higher number of magnets are two to three times more efficient and provide higher Lorentz force, well over the target of 10 $\mu$N (for example, for $N = 9$, $F_L^{\text{opt}} = 74.7$ $\mu$N). These ratios can be even further improved to maximize efficiency. For example, the maximum Lorentz force and efficiency $E_F$ were respectively 4.2 and 2.2 times higher for $N = 5$ than for $N = 1$ (see table 5), for system weights of 0.6 kg ($N = 5$) and 0.320 kg ($N = 1$). On the other hand, for a given weight, a quick saturation of $F_L^{\text{opt}}$ appears when increasing the number of magnets in Halbach arrays, as results obtained for $N = 9$ are only marginally better than those for $N = 5$. Our results suggest, however, that this saturation becomes less pronounced for heavier systems and therefore a higher number of magnets may be beneficial for less compact velocimeters than the ones we focused on in this paper.

A sensitivity analysis showed that the optimal design was practically unaffected by variations of fluid conductivity or flow rate. This result is highly favorable in view of applications where important variations of these quantities may be expected. The optimal design was, however, found to be very sensitive to the position of the system, relative to the duct. In particular, proximity to the duct is of paramount importance for optimal system performance. The sensitivity analysis also revealed that poor centering of the magnet system around the duct could degrade performance too. Most importantly, sensitivity to system position increases noticeably with $N$. Therefore a compromise may be imposed by the application’s environment and it may be wiser to reduce the number of magnets if precise enough positioning of the system cannot be achieved.

Acknowledgments

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