Time Machines, Maximal Extensions, and Zorn’s Lemma

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Abstract
We consider the use of Zorn’s lemma in establishing the existence of maximal extensions of space-times, and consider Krasnikov’s theorem [1] on the non-existence of time machines in the light of these considerations.

1 Introduction
This paper comprises some observations on the use of Zorn’s lemma and its application to the existence of maximal extensions in general relativity. We observe that some care is required in setting up a situation to which Zorn’s lemma must be applied, and that in general the satisfaction of the conditions of Zorn’s lemma is not automatic. We then apply these considerations to the arguments of Krasnikov against the existence of time machines [1], and the recent comments of Manchak [2] in this regard. We find that although Manchak’s argument shows that Krasnikov’s argument is incorrect as it stands, the argument is not as devastating as it first appears, and that an amended form of Krasnikov’s claim may still hold.

In the sequel, section 2 provides a brief review of the relevant aspects of Lorentz geometry and Zorn’s lemma, section 3 considers the application of Zorn’s lemma to proof of the existence of maximal extensions, section 4 provides some useful examples of when Zorn’s lemma does not apply, and section 5 considers the consequences for Krasnikov’s argument (and Manchak’s refutation).

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2 Background

First, we recall some basic ideas of Lorentz geometry.

A space-time is a pair \((M, g)\) where \(M\) is an \(n\)-dimensional smooth manifold without boundary, and \(g\) is a Lorentz metric, i.e. a smooth non-degenerate pseudo-Riemannian metric of signature \((+, -, \ldots, -)\) on \(M\). If \(\xi\) is a tangent vector at some point of \(M\), then we say that \(\xi\) is timelike if \(g(\xi, \xi) > 0\), null if \(g(\xi, \xi) = 0\) and spacelike if \(g(\xi, \xi) < 0\). We will assume that all space-times are time-orientable, meaning that they admit a smooth continuous timelike vector field. This is not a significant restriction, as any space-time which is not time-orientable has a time-orientable double cover [3]. We can therefore consistently talk about future- or past-directed timelike and null vectors.

If \(p, q \in M\) we say that \(q\) is in the chronological future of \(p\), written \(p \ll q\), if there is a smooth non-constant curve from \(p\) to \(q\) whose tangent vector at each point is timelike and future directed; we also define \(I^+(p) = \{ q \in M | p \ll q \}\). Similarly, \(q\) is in the causal future of \(p\), written \(p < q\) if there is a smooth curve from \(p\) to \(q\) whose tangent vector at each point is timelike or null and future directed, and we define \(J^+(p) = \{ q \in M | p < q \}\). \(I^-\) and \(J^-\) are defined by replacing “future directed” by “past directed” in the above.

There is a large (at least countably infinite [4]) family of constraints that can be placed on \(M\) in terms of these causal relationships—see [5] for a review of some of the more interesting ones. The condition of primary concern here is that the space-time be chronological: this means that there is no point \(p\) such that \(p \ll p\). A timelike curve connecting a point to itself is called a closed timelike curve, abbreviated to CTC, and CTCs are a prerequisite for any space-time to be regarded as containing a time machine. Whether such CTCs can arise in certain situations (or at all) is the topic of an extensive discussion in the literature: see the references in Manchak [2] for one thread of the argument.

Another important idea in the study of global structure is that of a maximal extension of a space-time. Given a space-time \((M, g)\), we say that \((M', g')\) is a maximal extension of \((M, g)\) if there is an isometry from \((M, g)\) to a subset of \((M', g')\), but no isometry from \((M', g')\) to a proper subset of any space-time.

A standard tool in demonstrating the existence of such an extension is Zorn’s lemma. As this is likely to be less familiar to the reader than the material on Lorentz geometry briefly summarised above, we give a complete account of the relevant definitions from the theory of partially ordered sets.

A partially ordered set (poset) consists of a set, \(Z\), and a relation \(\leq\) on \(Z\) such that for all \(a, b, c \in Z\),

1. \(a \leq a\)
2. \(a \leq b\) and \(b \leq a\) implies \(a = b\)
3. \(a \leq b\) and \(b \leq c\) implies \(a \leq c\).

\(a \in Z\) is an upper bound of \(V \subseteq Z\) if \(b \leq a\) for all \(b \in V\). A chain in \(Z\) is a subset \(V\) such that for any \(a, b \in V\), either \(a \leq b\) or \(b \leq a\). Finally, a maximal element \(a \in Z\)
is an element such that \( a \leq b \) implies \( a = b \). (Note: \( a \) need not exceed all elements of \( Z \); it is only necessary that no element of \( Z \) exceed \( a \). Thus a poset \( Z \) may have many maximal elements.)

**Theorem 1** Zorn’s Lemma: let \((Z, \leq)\) be a poset set such that any chain in \( Z \) has an upper bound. Then \( Z \) has a maximal element. \( \square \)

See any text on set theory, eg [6] for a fuller discussion of this, and an account of its relationship with the axiom of choice.

### 3 Zorn’s Lemma and Maximal Extensions

The first step is, naturally, to fix our poset so that we can use Zorn’s lemma to argue about the existence of maximal extensions.

The obvious starting point is to consider the set of isometry classes of Lorentz manifolds, with the partial ordering \( \leq \) given by \((M, g_M) \leq (N, g_N)\) if there is an isometry from \((M, g_M)\) to a submanifold of \((N, g_N)\). Unfortunately, this does not place a partial ordering on the set of all Lorentz manifolds, as it is possible to find \((M, g_M)\) and \((N, g_N)\) such that \((M, g_M) \leq (N, g_N)\) and \((N, g_N) \leq (M, g_M)\), while \((M, g_M)\) and \((N, g_N)\) are not isometric.

**Example 1** Let \( C_1 \) be the curve in \( \mathbb{R}^3 \) given \((2 + \cos(t), 0, t)\) for \( t > 0 \), and \( C_2 \) be the curve given by \((2 + \sin(t), 0, t)\) for \( t > 0 \). Let \( S_i \ (i=1,2) \) be the surface given by rotating \( C_i \) about the \( z \)-axis, and equipped with the induced metric. Then clearly \( S_1 \) is isometric to a proper subset of \( S_2 \), and \( S_2 \) is isometric to a proper subset of \( S_1 \), while \( S_1 \) and \( S_2 \) are not isometric. Taking the product of \( S_1 \) and \( S_2 \) with a timelike line yields the required pair of Lorentz manifolds.

However, we are interested in the existence of a maximal extension of a particular space-time, say \((M, g)\). We can therefore restrict our attention to the set of isometry classes of Lorentz manifolds \((M_\alpha, g_\alpha)\) such that there is an isometry \( f_\alpha \) with domain \((M, g)\) and codomain a submanifold of \((M_\alpha, g_\alpha)\), with a fixed choice (if there is any) of \( f_\alpha \). The partial ordering is then given by \((M_\alpha, g_\alpha) \leq (M_\beta, g_\beta)\) if there is an isometric inclusion, i.e. an isometry \( i \) from \((M_\alpha, g_\alpha)\) to a submanifold of \((M_\beta, g_\beta)\) such that \( f_\beta = i \circ f_\alpha \). But any two isometries which agree on an open set must in fact be equal: this follows from the stronger result that local isometry is determined by its differential map at a single point (Proposition 62 of Chapter 3 of [7]). It follows that if \((M_\alpha, g_\alpha) \leq (M_\beta, g_\beta)\) and \((M_\beta, g_\beta) \leq (M_\alpha, g_\alpha)\), then \((M_\alpha, g_\alpha) = (M_\beta, g_\beta)\), and so we have a poset.

With this context, we can now see almost immediately that any Lorentz manifold has a maximal extension. If \((M_\alpha, g_\alpha)\) is an increasing chain, then \( \cup M_\alpha \) provides an upper bound, and so Zorn’s lemma can be applied. In general, of course, \((M, g)\) will have many inequivalent extensions.
4 Maximal extensions of locally constrained Lorentz manifolds

In practice, if the space-time $M$ satisfies some causal constraint such as begin chronological, or globally hyperbolic, we are interested in establishing the existence of a maximal extension which satisfies the same constraint. In particular, we need to consider the existence of maximal extensions satisfying the constraints classified as local by Krasnikov; to this end let us now recall Krasnikov’s notion of a local condition [1].

**Definition 1** Condition $C$ is local if the following is true: For all space-times $(M, g)$, $C$ holds in $(M, g)$ if and only if, for all open proper subsets $U$ of $M$, $C$ holds in $(U, g|_U)$.

The definition as given by Krasnikov [1] does not in fact explicitly require that $U$ be a proper subset of $M$. However, without this requirement, the definition simply states that a condition is local if it holds in all open proper subsets of $M$, and it seems most likely from the context that the above formulation is what is intended. In any case, any condition which is local according to this definition is also local according to the strict reading of Krasnikov’s definition, so our conclusions will not be adversely affected by this slight change.

It is worth noting, however, that this definition has some unsatisfactory aspects: in particular, there are conditions which one would not be happy to call local which satisfy this definition—we will see below that stable causality is local according to Krasnikov’s definition. However, rather than introduce a new term for this condition, I will henceforth use 'Krasnikov-local', to retain contact with the previous literature.

We can then restrict our attention to space-times satisfying some Krasnikov-local condition $C$, with the following natural definitions.

**Definition 2** A space-time $(M, g)$ satisfying the condition $C$ is called a $C$-space-time. We then say that an extension of $(M, g)$ satisfying $C$ is a $C$-extension, and a $C$-extension of $(M, g)$ which is not isometric to a proper subset of any $C$-space-time is a $C$-maximal $C$-extension.

With this terminology in place, Krasnikov [1] asserts that it follows from Zorn’s lemma that any $C$-space-time has a $C$-maximal $C$-extension. His principal result is the no-go claim:

**Theorem 2** (Krasnikov) Any chronological $C$-space-time $M$ has a $C$-maximal $C$-extension whose CTCs are confined to $I^−(M)$.

This assertion is also used by Manchak [2] in his counter-argument to this claim.

The argument from Zorn’s lemma certainly works in at least some cases. First, we note that the chronological condition is Krasnikov-local. If $(M, g)$ is chronological and we consider all chronological space-times containing $(M, g)$, it is easy to see that every chain has a chronological upper bound, and so we can immediately deduce the
existence of maximal chronological extensions to any chronological space-time. However, the situation is not so clear in other cases.

First, we will examine a causality condition which is often imposed as a version of what it means for a space-time to be physically reasonably, and we will see that this condition is Krasnikov-local, but which does not allow the application of Zorn’s lemma. This condition is stable causality [8].

Recall [9] that if $g$ and $g'$ are metrics on $M$, we say $g < g'$ if $g(\xi, \xi) \leq 0$ implies that $g'(\xi, \xi) < 0$ for all tangent vectors $\xi$. Then $(M, g)$ is said to be stably causal if there is a metric $g'$ such that $g < g'$ and $(M, g')$ has no CTCs.

We begin by showing that stable causality is a Krasnikov-local condition. So let $(M, g)$ be a stably causal space-time. It is clear that if $U$ is a proper subset of $M$, then $(U, g|_U)$ is stably causal.

Conversely, suppose that $(M, g)$ is not stably causal but that $(U, g|_U)$ is stably causal for any $U$ a proper subset of $M$. Fix some $p \in M$, and let $U = M \setminus \{p\}$. Let $g'$ be any metric such that $g < g'$, and let $\gamma$ be a CTC in $(M, g')$. If $p$ does not lie on $\gamma$, then $\gamma$ is a CTC in $(U, g|_U)$. If $p$ does lie on $\gamma$, then since $\gamma$ is timelike we can perturb $\gamma$ slightly in $\gamma'$, a CTC which avoids $p$; thus again $(U, g|_U)$ admits a CTC. This is true no matter which $g'$ is chosen, and so $(U, g|_U)$ is not stably causal.

We therefore see

**Theorem 3** The condition of stable causality is Krasnikov-local.

But now, let $\mathbb{M}^2$ be 2-dimensional Minkowski space, with the usual coordinates $(x, t)$ and metric $dt^2 - dx^2$, let $M$ be the space-time obtained from the region of $\mathbb{M}^2$ with $-2 \leq t \leq 2$, minus the line $\{(x, 0)|x \leq 0\}$, and with $(x, 2)$ identified with $(x, -2)$ for all $x$. We now define the family of space-times $M_n$ by

$$M_n = M \setminus \{(x, 1)|x > -1 - 1/n\} \cup \{(x, -1)|x > -1 - 1/n\}. $$

Clearly, the $M_n$ form a chain, since if $m < n$ it follows that $M_m \subset M_n$ with the obvious isometric inclusion. But any upper bound of this chain must include $\bigcup M_n$, and this union is not stably causal: it is the example on p 197 of [8]. Hence this chain does not have a stably causal upper bound, and so Zorn’s lemma cannot be applied to show that a stably causal space-time has a stably causal maximal extension.

It is important to note that this does not imply that none of the $M_n$ has a stably causal maximal extension. In fact, it is not difficult to construct such an extension. We define $M_n^+$, $M_n^-$ and $M_n^0$ by

$$M_n^\pm = \mathbb{M} \setminus \{(x, \pm 1)|x > -1 - 1/n\} $$
$$M_n^0 = \mathbb{M} \setminus \{(x, 0)|x \leq 0\} $$

Then we attach the upper edge of the slit of $M_n^+$ to the lower edge of the upper slit in $M_n$, and the lower edge of the slit in $M_n^+$ to the upper edge of the upper slit in $M_n$; similarly with $M^0$ and the middle slit in $M_n$, and with $M^-$ and the lower slit in $M_n$: 5
this construction is similar to that in Section 3.1 of [8]. The resulting space is clearly maximal and stably causal.

We therefore see that we cannot always apply Zorn’s lemma to deduce that a C-space-time must have a C-maximal C-extension, and must at least sometimes deduce the existence by other means. Nevertheless, it might still be the case that the required maximal extension always exists, and an alternative argument would remove this difficulty.

But in fact, we can show much more:

**Theorem 4** There is a Krasnikov-local condition \( \mathcal{V} \) such that no \( \mathcal{V} \)-space-time has a \( \mathcal{V} \)-maximal \( \mathcal{V} \)-extension.

First, recall the definition of almost maximal from Manchak [2].

**Definition 3** A space-time \((M, g)\) is almost maximal if it has a maximal extension \((M', g')\) such that \(M' \setminus M\) is a finite set.

Then our condition \( \mathcal{V} \) is that \((M, g)\) be neither maximal nor almost maximal.

First, we observe that any proper open subset of a \( \mathcal{V} \)-space-time is a \( \mathcal{V} \)-space-time. Next, suppose that for any proper open subset \( U \) of \((M, g)\), the space-time \((U, g|_U)\) is a \( \mathcal{V} \)-space-time. Then, in particular, for any \( U = M \setminus \{p\} \) where \( p \in M \), \((U, g|_U)\) is a \( \mathcal{V} \)-space-time. It follows immediately that \((M, g)\) satisfies condition \( \mathcal{V} \). Thus \( \mathcal{V} \) is a Krasnikov-local condition.

So let \((M, g)\) be a \( \mathcal{V} \)-space-time. Then \((M, g)\) is not maximal, and every maximal extension \((M', g')\) is such that \(M' \setminus M\) is infinite. Therefore \((M, g)\) has a \( \mathcal{V} \)-extension obtained by removing infinitely many, but not all points of \(M' \setminus M\) from \(M'\). Since any \( \mathcal{V} \)-space-time has a proper \( \mathcal{V} \)-extension, no \( \mathcal{V} \)-space-time has a \( \mathcal{V} \)-maximal \( \mathcal{V} \)-extension.

Therefore if we wish to make use of the properties of a C-maximal C-extension of a C-space-time, we must either first check that any chain of C-space-times satisfying the condition has an upper bound satisfying condition C, or establish the existence of a C-maximal C-extension by some other means.

## 5 No no-go?

First, we briefly recall Manchak’s counter-argument [2] to Krasnikov’s theorem.

**Definition 4** A space-time satisfies condition \( \mathcal{U} \) if it satisfies one of the following: (i) is is either maximal or almost maximal and contains CTCs or (ii) it is neither maximal nor almost maximal.

Then condition \( \mathcal{U} \) is Krasnikov-local, and any \( \mathcal{U} \)-maximal space-time contains CTCs.

Now let \( N \) be the manifold \( \mathbb{R} \times S^1 \) with coordinates \((t, \phi)\), and let \( g \) be the metric given by

\[
\text{ds}^2 = 2dt d\phi + t d\phi^2
\]
Next, consider \((M, g)\) where \(M\) is the region of \(N\) with \(t < 0\), and \(g\) is the induced metric on this submanifold. This space-time is clearly neither maximal nor nearly maximal, and so satisfies condition \(\mathcal{U}\). It follows that any \(\mathcal{U}\)-maximal \(\mathcal{U}\)-extension therefore contains CTC’s, and these cannot lie in the region \(I^-(M)\), since in any extension \(I^-(M) = M\). This provides a contradiction to Krasnikov’s claim that any \(C\)-space-time \(M\) has a \(C\)-maximal \(C\)-extension whose CTCs are confined to \(I^-(M)\).

But this counter-example requires the existence of a \(\mathcal{U}\)-maximal \(\mathcal{U}\)-extension to a \(\mathcal{U}\) space-time, which has not yet been established, as it has not been shown that the conditions of Zorn’s lemma are satisfied. In fact, they are not.

In order to see this, define
\[
M_n = M^2 \setminus \{(m^2, 0)|m \in \mathbb{Z}, m \geq n\}.
\]
for each positive integer \(n\).

Clearly, each \(M_n\) satisfies \(\mathcal{U}\), and if \(n_1 < n_2\), \(M_{n_1} \subset M_{n_2}\) via the obvious isometric inclusion. But \(\bigcup M_n = M^2\), which is maximal, but does not contain CTCs. So we have a chain, \(M_n\), of \(\mathcal{U}\)-space-times, which does not have an upper bound satisfying condition \(\mathcal{U}\). Hence we cannot deduce the existence of \(\mathcal{U}\)-maximal extensions from Zorn’s lemma, and thus the existence argument for a counter-example to Krasnikov’s claim is incomplete.

This bears closer inspection. Krasnikov’s claim may be usefully divided into two parts: first, every \(C\)-space-time \((M, g)\) has a \(C\)-maximal \(C\)-extension, \((M', g')\) and second, this extension has all its CTCs in the past of \(M\). Then Manchak’s example shows that under the assumption that the first part is true, the second part need not be. However, we know that some \(C\)-space-times do not admit a \(C\)-maximal \(C\)-extension.

It therefore remains logically possible that every \(C\)-space-time \((M, g)\) which has a \(C\)-maximal \(C\)-extension has such an extension with all CTCs to the past of \(M\), but that the subset of Misner space considered by Manchak has no \(\mathcal{U}\)-maximal \(\mathcal{U}\)-extension, and so does not provide a counter-example to this modified statement. To exclude this possibility requires either a proof that the required \(\mathcal{U}\)-maximal \(\mathcal{U}\)-extension exists, or a new counter-example.

6 Conclusions

A degree of care is required when asserting the existence of maximal extensions satisfying constraints. Since Krasnikov’s proof assumes the existence of \(C\)-maximal \(C\)-extensions whenever \(C\) is a Krasnikov-local condition, his claim is certainly incorrect as it stands. A modified claim restricting attention to those space-times with the required maximal extension (and establishing that such a restriction is physically reasonable) is not yet disproved by Manchak’s counter-example, since it makes use of the same assumption.

References


