A classical conundrum

or

the importance of asking the right question.

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Abstract
We examine an apparent paradox in probability theory, and conclude that care must be taken in framing questions.

1 The conundrum

There are two boxes in front of you; you are told that each box contains a positive amount of money, and one box contains ten times as much as the other. Upon opening one of the boxes, you find inside a ten pound note. Contemplating the situation, you reason as follows. The other box contains either one hundred pounds, or one pound, with equal likelihood. If you switch boxes, you will raise your expectation of the amount of money you acquire from ten pounds to fifty pounds and fifty pence.

At this point a rather unpleasant thought strikes you. Whichever box you open and however much money there is in it, you can reason that way, and deduce that switching boxes is the best strategy. Indeed, you can argue this before you open either box! Upon reaching out your hand to open either
box, you can argue that you will, on average, be better off with the other box. You are now in a rather worse position than Buridan’s ass, since each box seems likely to give a greater reward than the other.

How can this be?

2 The resolution

Let us agree that arguments based on the discreteness of our monetary system are illusive, and that the amount of money in either box may be any positive real number, subject only to the constraint that one box contain ten times as much as the other. Then what is the actual situation under discussion?

A technical restatement of the problem is as follows. We have two boxes, \( A \) and \( B \). Let \( f(x,y)dx\,dy \) be the probability that the amount of money in box \( A \) is in \((x, x + dx)\), and that in box \( B \) is in \((y, y + dy)\). Then it must be that for any \((x, y)\), \( f(x,10y) = f(x,y/10) \), and this latter quantity is in fact zero if \( x \neq y \); \( f \) is thus concentrated on the lines \( y = 10x \) and \( y = x/10 \).

Now, we must have

\[
\int_0^\infty \int_0^\infty f(x,y)dx\,dy = 1
\]

since \( f \) is a probability measure, and furthermore,

\[
\int_0^\infty \int_0^\infty f(x,10y)dx\,dy = \int_0^\infty \int_0^\infty f(x,y/10)dx\,dy
\]

But by a simple change of variable, we see that

\[
\int_0^\infty \int_0^\infty f(x,10y)dx\,dy = 1/10
\]

and

\[
\int_0^\infty \int_0^\infty f(x,y/10)dx\,dy = 10
\]

It is clear that no measure, \( f \), can have this property. The resolution of the apparent paradox is, then, that the situation cannot be set up.
3 Discussion and explanation

At once you may be motivated to retort, “But wait; I pick an amount of money at random, and put it in one box. Then I toss a coin and depending on the outcome of that toss, either put ten times as much money or a tenth as much money into the other.”

However, the initial amount of money must have been picked according to some non-uniform probability density. Given this density, the amount of money in the box opened gives partial information as to whether the amount in the other box will be ten times as much, or a tenth as much.

We thus see that, although the question can be phrased in a plausible manner, it cannot really be asked at all, since it involves assumptions on the means of placing the money in the boxes that cannot be satisfied.

4 Moral

It is worth examining the statement of an apparently intractable problem in order to make sure that there really is a problem there to solve.

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