Visualizing Light Cones in Schwarzschild Space

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Abstract

Although introductory courses in special relativity give an introduction to the causal structure of Minkowski space, it is common for causal structure in general space-times to be regarded as an advanced topic, and omitted from introductory courses in general relativity, although the related topic of gravitational lensing is often included.

Here a numerical approach to visualizing the light cones in exterior Schwarzschild space is used to make some of these ideas accessible in the context of a first course in general relativity.

1 Introduction

Purpose of paper to provide paedagogical introduction to causal structure via consideration of Schwarzschild space, making use of numerical methods to visualize light cones, and forming a bridge between an introductory course at level of, for example, d’Inveno or Hughston and Tod, and the more advanced treatment of Wald or Hawking and Ellis. Should be accessible to a student who has taken an undergraduate course in GR which includes an introduction to the Schwarzschild solution.

Overview. Section 1, causal structure from Minkowski to general space-times. Importance of null geodesics. Section 2, special properties of null geodesics in static space-times.

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2 Causal Structure

We begin by reviewing the causal structure of Minkowski space-time, and seeing how these ideas work in the case of a general space-time.

So let $\mathbb{M}$ be Minkowski space, with the standard coordinates $(x, y, z, t) = (x^1, x^2, x^3, x^4)$, where we work in units in which $c = 1$. We will $x^a$ as an abbreviation $(x^1, x^2, x^3, x^4)$, with $a$ being understood to take the values 1, 2, 3, 4.

Given two points, $P, Q \in \mathbb{M}$, with coordinates $(x_P, y_P, z_P, t_P)$ and $(x_Q, y_Q, z_Q, t_Q)$, the vector connecting $P$ to $Q$ is

$$\vec{PQ} = (x_Q - x_P, y_Q - y_P, z_Q - z_P, t_Q - t_P).$$

we see that the straight line connecting $P$ to $Q$ corresponds to the trajectory of a particle travelling with constant velocity, which is travelling slower than light if

$$(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2 < (t_Q - t_P)^2,$$

at the speed of light if

$$(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2 = (t_Q - t_P)^2,$$

and faster than light if

$$(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2 > (t_Q - t_P)^2.$$

Only the first two possibilities are observed experimentally, and since the third possibility implies the ability to send a message into one’s past, we exclude it as the possible trajectory of a physical particle.

In the first case, then, if $t_Q > t_P$, it is possible to send a massive particle from $P$ to $Q$. We say that $Q$ is to the chronological future of $P$, and write $Q \in I^+(P)$. We also define $I^-$ by stating that this is equivalent to $P \in I^-(Q)$, and $I(P)$ is the set of all points chronologically connected to $P$, i.e. it is $I^+(P) \cup I^-(P)$.

In the second case, if $t_Q > t_P$, it is possible to send a massless particle, e.g. a photon, from $P$ to $Q$. The set of events which are connected to $P$ by such a vector is called the light cone of $P$, and can be split into a future part (with time coordinate greater than $t_P$) and a past part (with time coordinate less than $t_P$). The light cone of $P$ is the boundary of $I(P)$. The interior of this cone comprises the chronological future and past of $P$.

This is very familiar from introductory courses in special relativity, and lucid descriptions can be found in the standard texts, such as Rindler or Taylor. However, this approach does not easily extend to more general space-times,
so we begin by considering a different, but equivalent, way of characterizing the future and past of an event.

To this end, let $x^a(s)$ be a curve in $\mathbb{M}$, parameterised by $s$. Then the tangent vector to this curve is

$$v^a(s) = \frac{dx^a}{ds} = \dot{x}^a$$

and we say that this tangent is timelike, null, or spacelike depending on whether

$$(v^1)^2 + (v^2)^2 + (v^3)^2 - (v^4)^2$$

is negative, zero, or positive. If it is timelike, we say that it is future pointing if $v^4 > 0$ and past pointing if $v^4 < 0$.

Then it is fairly obvious, and not too difficult to prove, that $Q \in I^+(P)$ if and only if there is a curve which starts at $P$, ends at $Q$, and has a future pointing timelike tangent vector at each point; and that $Q$ lies on the future part of the null cone if there is a curve starting at $P$ and ending at $Q$ whose tangent is future pointing and null, but no such curve whose tangent is future pointing and timelike.

This is the characterization which we carry over to a general space-time.

So if $M$ is a space-time with metric $g_{ab}$, we have a null cone at each point consisting of those vectors $n^a$ such that $g_{ab}n^a n^b = 0$. At each point, this cone splits into two halves: if it is possible to choose one of the two consistently over the whole of $M$, we say that $M$ is time orientable, and choose one of these two halves to be the future half.

Then given a curve $x^a(s)$, we can, just as before, consider its tangent vector

$$v^a = \frac{dx^a}{ds} = \dot{x}^a$$

and define it to be timelike, null, or spacelike depending on whether $g_{ab}v^a v^b$ is negative, zero, or positive; and if it is timelike, future pointing if it points into the future half of the null cone, and past pointing if it points into the past half.

Given two events, $P$ and $Q$ in $M$, we then say that $Q \in I^+(P)$ if there is a curve starting at $P$ and ending at $Q$, whose tangent vector is everywhere timelike future pointing, and similarly for $I^-(P)$.

This relationship defines the causal structure of space-time, because it describes the pairs of events, $P$ and $Q$, such that a signal can propagate from $P$ to $Q$. In other words, it tells us when something happening at $P$ can influence the state at $Q$. This is clearly a fundamental property of space-time, and given an event, $P$, one would like to know just what $I^+(P)$ is.
The situation is similar to that of Minkowski space-time, but not identical. It is important to note that in Minkowski space-time, \( Q \in I^+(P) \) if and only if there is a future-pointing geodesic connecting \( P \) to \( Q \), but this is not true in general, though it is true if \( Q \) is near enough to \( P \).

More importantly, though, there are differences between the light cone through \( P \) and the boundary of \( I^+(P) \). In general, there can be points on the boundary of \( I^+(P) \) which are not connected to \( P \) by a null geodesic, and points on future-pointing null geodesics through \( P \) which are not on the boundary of \( I^+(P) \).

We will not be concerned with the first possibility, as it does not occur in the type of space-time we will consider. However, the second possibility generally occurs in space-times with a non-flat metric, since in this case the null geodesics emanating from an event, \( P \), can subsequently intersect because of gravitational focussing, and points further along a null geodesic than such a point of intersection lie inside \( I^+(P) \).

In the next two sections we will see how to describe the null cones of a family of space-times which include Schwarzschild space-time, and also produce diagrams obtained by a numerical investigation of the null geodesics.

3 Static Space-Times

Remember that a static space-time is one in which we have a coordinate system such that the metric takes the form

\[
g_{\alpha\beta}dx^\alpha dx^\beta - f(x^\alpha)dt^2
\]

where \( \alpha = 1, 2, 3 \), and \( f(x^\alpha) > 0 \).

One case of particular interest is Schwarzschild space-time, which in the usual coordinate system has the metric

\[
\frac{1}{(1 - \frac{2M}{r})^2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \left(1 - \frac{2M}{r}\right)dt^2.
\]

where we consider only the region \( r > 2M \), which is the region outside the black hole.

First, we will derive a useful property of null geodesics in general, and see how to use this to simplify the problem of finding the null geodesics in static space-times.

In a general space-time, \( M \), with metric \( g_{ab} \), an affinely parameterised null geodesic is a curve \( x^a(s) \) such that \( g_{ab}\dot{x}^a\dot{x}^b = 0 \) and satisfying the differential equation

\[
\ddot{x}^a(s) + \Gamma^a_{bc}\dot{x}^b(s)\dot{x}^c(s) = 0,
\]
where $\Gamma_{bc}^a$ are the Christoffel symbols.

Now, suppose we replace the metric $g_{ab}$ by the metric $\tilde{g}_{ab} = e^\Omega g_{ab}$, where $\Omega$ is an arbitrary smooth (real-valued) function of $G$. If we denote by $\Gamma_{bc}^a$ and $\tilde{\Gamma}_{bc}^a$ the Christoffel symbols of $g_{ab}$ and $\tilde{g}_{ab}$ respectively, then, denoting $\frac{\partial F}{\partial x^a}$ by $F_a$, for any quantity $F$, we obtain

$$\tilde{\Gamma}_{bc}^a = \frac{1}{2} \tilde{g}^{ad} \{ \tilde{g}_{cd,b} + \tilde{g}_{bd,c} - \tilde{g}_{bc,d} \} = \frac{1}{2} e^{-\Omega} g^{ad} \{ (e^\Omega g_{cd})_b + (e^\Omega g_{bd})_c - (e^\Omega g_{bc})_d \} = \Gamma_{bc}^a + \frac{1}{2} g^{ad}(\Omega_b g_{cd} + \Omega_c g_{bd} - \Omega_d g_{bc}).$$

It follows that if

$$\ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = 0$$

then

$$\ddot{x}^a + \tilde{\Gamma}_{bc}^a \dot{x}^b \dot{x}^c = \ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c + \frac{1}{2} g^{ad}(\Omega_b g_{cd} + \Omega_c g_{bd} - \Omega_d g_{bc}) \dot{x}^b \dot{x}^c = \ddot{x}^a + \tilde{\Omega} \dot{x}^a = \tilde{\Omega} \dot{x}^a$$

since $\dot{x}^a$ is null.

But this is just the equation of a null geodesic which is not parametrised by an affine parameter. So null geodesics are in fact invariant under conformal transformations, although the affine parameters are different.

We can now simplify the problem of describing the null geodesics of a static space-time. We know that the null geodesics of the metric $g_{\alpha\beta} dx^\alpha dx^\beta - f(x^\alpha) dt^2$ are the same as those of

$$\tilde{g}_{\alpha\beta} dx^\alpha dx^\beta - dt^2$$

where $\tilde{g}_{\alpha\beta} = g_{\alpha\beta}/f$, so we can find the null geodesics of the original space-time by finding the null geodesics of this conformally related one.

But the geodesic equations in the second case become

$$\dot{t} = 0$$

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the three dimensional Riemannian metric $g_{\alpha\beta}$. 
We can now immediately see that if $(x^\alpha(t), t)$ is a null geodesic in $M$ parameterised by $t$ (which will not, in general, be an affine parameter), then $x^\alpha(t)$ will be an arc-length parameterised geodesic of the Riemannian metric $\tilde{g}_{\alpha\beta}$, and vice versa. We can therefore visualise the null geodesics in $M$ by finding the arc-length parameterized geodesics of $\tilde{g}_{\alpha\beta}$ and assigning the appropriate $t$ value to each point.

4 Schwarzschild Space-Time

We have the metric
\[
\frac{1}{(1 - \frac{2M}{r})^2} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \left(1 - \frac{2M}{r}\right) dt^2.
\]
which we immediately replace by the conformally related metric
\[
\frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{1 - \frac{2M}{r}}(d\theta^2 + \sin^2 \theta d\phi^2) - dt^2
\]
and from which we extract the Riemannian metric
\[
\frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{1 - \frac{2M}{r}}(d\theta^2 + \sin^2 \theta d\phi^2).
\]
We can then find geodesics of this second metric $(r(t), \theta(a), \phi(t), t)$, where $t$ is the arc-length parameter of the geodesics of the third metric.

Use of the symmetry of Schwarzschild space-time to reduce the problem by one dimension.

5 Visualizing the Light Cones

Use of MATLAB to solve equations and give interactive pictures of the light cones.

6 Conclusion

Although some of the subtler mathematics is glossed over, can begin to see how causal structure of general space-times is different from Minkowski space.
7 Old Paper

Within the frameworks of special and general relativity, it is generally taken as an axiom that no material influence can travel faster than light. The reason for this is that such a propagation necessarily allows time-travel, which can lead to severe conceptual difficulties [1]. There have been attempts to resolve such paradoxes within the context of quantum field theory [2], but we will restrict our attention to the classical regime, where the axiom is a useful one.

We say that the space-time point \( P \) is causally related, or causally connected to the space-time point \( Q \) if it is possible for a signal to pass from \( P \) to \( Q \) at a speed no greater than that of light, in other words if a material particle can travel from one to the other. Thus, two points are causally connected if the values of physical fields at one can affect the values of physical fields at the other. Related notions of are of particular relevance to much of the structure of general relativity, and particularly the singularity theorems, which predict that under certain reasonable conditions the gravitational field equations must break down [3, 4], in the sense that if we want to be able to predict the evolution of the universe from data on an initial surface, there are free-fall trajectories along which curvature scalars diverge in finite affine parameter time.

Now, in Minkowski space it is particularly easy to find out whether two points \( P \) and \( Q \) are causally connected. One simply considers the vector connecting them: if it is a timelike or null vector, then the points are causally connected, otherwise they are not. In addition, there is a good geometrical way of seeing whether or not points are causally connected. One simply considers some surface of constant time containing one of the points—say \( P \)—and then considers where the light cone of the other, \( Q \), cuts this surface. It will cut the surface in a sphere \( \Sigma \), and \( P \) and \( Q \) are causally connected if \( P \) lies inside or on \( \Sigma \).

Unfortunately, this pleasant picture is rather less useful in general relativity, since (a) points are no longer connected by vectors, and (b) light cones can behave in distinctly unfriendly ways. Even when it is still possible to construct surfaces of constant time, the intersection of a light cone with the surface will no longer be a smooth sphere with no self-intersections, in general. However, in the special case of the exterior portion of Schwarzschild space-time, it turns out that we can largely recover this geometric picture, although the algebraic one is still very complicated.

It is also the case that, thanks to the various symmetry properties of this metric, and with the aid of a little judicious numerical solving of differential equations, it is possible to obtain pictures of how the light cone of a point intersects a surface of constant \( t \) in this space-time.
In section 8, we recall how to use a variational principle to obtain the equations describing geodesics in a general space-time, and see that if two metrics $g_{ab}$ and $\tilde{g}_{ab}$ are conformally related—i.e. $\tilde{g}_{ab} = e^{\Omega} g_{ab}$, where $\Omega$ is some smooth function on the space-time—then they have the same null geodesics. In section 9, we will specialize to the Schwarzschild metric,

$$\text{d}s^2 = -(1 - \frac{2M}{r})\text{d}t^2 + \frac{1}{1 - \frac{2M}{r}} \text{d}r^2 + r^2(\text{d}\theta^2 + \sin^2(\theta)\text{d}\phi^2)$$

and restrict our attention to the region $r > 2M$, which we will refer to as $M$. (Note that here, and throughout, we use geometric units, so that $c = G = 1$.) There, we will see how to reduce the problem of constructing the surfaces in which light cones intersect slices of constant $t$ to that of finding geodesics in a 2-dimensional space. In section 10 we will consider diagrams obtained by numerical integration of the geodesic equations, which can be used as an aid to visualizing the light cones in Schwarzschild space, and hence help us to see whether two points are in fact causally related. This will, incidentally, provide a way of seeing how gravitational sources can behave as lenses. Finally, we will consider the limitations of this approach to visualizing the causal structure of a space-time. The paper should be accessible to an advanced undergraduate or beginning postgraduate student who has taken an introductory course in general relativity.

8 Some generalities on geodesics

In a general space-time, $G$, with metric $g_{ab}$, we can obtain the equations defining a geodesic by means of a variational principle. If $x^a(p)$, a curve in $G$, is given by

$$\delta \int g_{ab}\dot{x}^a\dot{x}^b\text{d}p = 0,$$

then $x^a(p)$ is a geodesic, parameterised by arc length, except for the special case when $||\dot{x}^a(p)|| = 0$, when $p$ is an affine parameter. In either case, the Euler-Lagrange equations are equivalent [8] to

$$\ddot{x}^a(p) + \Gamma^a_{bc}\dot{x}^b(p)\dot{x}^c(p) = 0.$$

Now, suppose we replace the metric $g_{ab}$ by the metric $\tilde{g}_{ab} = e^{\Omega} g_{ab}$, where $\Omega$ is an arbitrary smooth (real-valued) function of $G$. If we denote by $\Gamma^a_{bc}$ and $\tilde{\Gamma}^a_{bc}$ the Christoffel symbols of $g_{ab}$ and $\tilde{g}_{ab}$ respectively, then, denoting $\frac{\partial F}{\partial x^a}$ by
for any quantity $F$, we obtain

$$\tilde{\Gamma}^a_{bc} = \frac{1}{2} \tilde{g}^{ad} \{ \tilde{g}_{cd, b} + \tilde{g}_{bd, c} - \tilde{g}_{bc, d} \}$$

$$= \frac{1}{2} e^{-\Omega} g^{ad} \left\{ (e^{\Omega} g_{cd})_b + (e^{\Omega} g_{bd})_c - (e^{\Omega} g_{bc})_d \right\}$$

$$= \Gamma^a_{bc} + \frac{1}{2} g^{ad} (\Omega_{,b} g_{cd} + \Omega_{,c} g_{bd} - \Omega_{,d} g_{bc}).$$

It follows that if

$$\ddot{x}^a + \tilde{\Gamma}^a_{bc} \dot{x}^b \dot{x}^c = 0$$

then

$$\ddot{x}^a + \dot{\tilde{\Gamma}}^a_{bc} \dot{x}^b \dot{x}^c = \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c + \frac{1}{2} g^{ad} (\Omega_{,b} g_{cd} + \Omega_{,c} g_{bd} - \Omega_{,d} g_{bc}) \dot{x}^b \dot{x}^c$$

$$= \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c + \dot{\Omega} \dot{x}^a$$

$$= \dot{\Omega} \dot{x}^a$$

since $\dot{x}^a$ is null. But this is just the equation of a geodesic which is not parametrised by arc-length. So null geodesics are in fact invariant under conformal transformations, although the affine parameters are different. We will use this fact below when we change from the Schwarzschild metric to a more convenient one.

9 The Schwarzschild metric

It is part of the standard presentation of the study of motion in Schwarzschild space that one can reduce the geodesic equations to the case $\theta = \pi/2$, and solve for $r$ in terms of $\phi$ to obtain spatial trajectories for the geodesics [5, 6, 7].

This procedure provides many valuable insights into the geometry of Schwarzschild space, but is not adequate for the question we are addressing here; we need to know not only the spatial trajectory of a light ray, but also the trajectory as a function of time.

To do this, we use the property that null geodesics are unchanged by a conformal transformation of the metric, and consider the new metric

$$d\sigma^2 = -d\tau^2 + \frac{1}{(1 - 2M/r)^2} dr^2 + \frac{r^2}{1 - 2M/r} (d\theta^2 + \sin^2(\theta) d\phi^2)$$

Since this metric is related to the Schwarzschild metric by a conformal transformation, it has the same null geodesics; since we have made no coordinate
transformation, the surfaces of constant $t$ are still described by the same equation, namely $t = \text{constant}$.

Just as in the Schwarzschild case, geodesics that start off in and tangent to the equatorial subspace $\theta = \pi/2$ remain there; we can therefore restrict our attention to geodesics which satisfy this constraint and lie in the subspace defined by $\theta = \pi/2$. We will denote this subspace by $\mathcal{M}_{\pi/2}$. Later, of course, we must see how to reconstruct the full light cone of a point from its light cone restricted to this subspace. We thus consider the Euler-Lagrange equations derived from the metric

$$-dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2d\phi^2}{1 - \frac{2M}{r}}.$$

The $t$ equation is now simply $\ddot{t} = 0$, of which $t = p$ is a solution. With this, we see that the Euler-Lagrange equations for $r$ and $\phi$ are exactly those given by the variational principle

$$\delta \int \left( \frac{\dot{r}^2}{(1 - \frac{2M}{r})^2} + \frac{r^2\dot{\phi}^2}{1 - \frac{2M}{r}} \right) dt.$$

Thus if we have a null geodesic in our space-time, parameterised by the affine parameter $t$, its $r$ and $\phi$ coordinates give us a geodesic in the two-dimensional space with metric

$$\frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2d\phi^2}{1 - \frac{2M}{r}},$$

parameterised by arc-length.

The Euler-Lagrange equations in this case are

$$\ddot{\phi} = 2\frac{\dot{\phi}\dot{r}}{r} \left( \frac{3M - r}{r(r - 2M)} \right)$$

$$\ddot{r} = \dot{r}^2 \frac{2M}{r(r - 2M)} + \dot{\phi}^2 \left( \frac{r - 3M}{r} \right)$$

where the differentiation is now with respect to $t$.

Now, if we pick any point $P$, we can choose the coordinates so that $P$ lies at $(0, r(P), \pi/2, \phi(P))$, i.e. it involves no loss of generality to assume that $P$ lies in $\mathcal{M}_{\pi/2}$ with $t(P) = 0$. So we see that in $\mathcal{M}_{\pi/2}$, the set in which the light cone of $P$ intersects the surface of constant time $\{t = \tau\}$ is exactly that set of points obtained by starting at $(\tau, r(P), \phi(P))$ and travelling a geodesic distance $\tau \dot{r}$ along each possible geodesic in $\{t = \tau\}$. Let us call this set the equatorial $P$-wavefront at time $\tau$, and the full set in which the light cone of $P$ intersects $\{t = \tau\}$ simply the $P$-wavefront at time $\tau$. Alternatively, one
could say that the $P$-wavefront at time $\tau$ is the geodesic sphere of radius $\tau$, centred on $(\tau, r(P), \phi(P))$.

This is fine for giving us the picture in an equatorial subspace of $\mathcal{M}$, but what about the full picture? Well, there are in fact many possible choices of coordinates that place $P$ at $\theta = \pi/2$. For consider the three dimensional surface of constant time, $\{t = 0\}$, and the radial line in this surface containing $P$. Any of the two-dimensional planes in this surface containing this radial line may be taken as an equatorial plane by some choice of the spherical polar coordinates. And if we choose the same plane in each constant time surface, we obtain a version of $\mathcal{M}_{\pi/2}$. Any null geodesic must lie in some such equatorial subspace, and the suitable ones are obtained by rotating about the radial line containing $P$. Hence we can construct the $P$-wavefront at any time by finding the equatorial wavefront at that time and rotating the result. In the following section we will see some specific examples.

It is important to note that this only enables us to examine the causal relationship between a pair of points in the exterior portion of space-time, as the construction only allows us to consider portions of space-time with finite $t$-coordinate. Although the space-time may be analytically continued through the horizon by means of new coordinates, this cannot be done in such a way as to leave $t$ finite, and so we are unable to extend the discussion to include the region inside the event horizon.

10 Visualizing the light cones

Now, the geodesic equations are rather intractable by analytic methods, and require the application of numerical techniques. To simplify the calculations, we can assume that $\phi(P) = 0$, so that we only have to consider the effect of starting off at different $r$-values. Given an initial value for $r$, we need to consider various directions a geodesic could leave in, evolve each for the same time, and then connect up the end-points in the appropriate order. This will give us one slice of the $P$-wavefront at time $t$, which we must mentally rotate to recover the true picture.

On first consideration, there are three basic regions that $P$ may be in; inside the photon sphere, on the photon sphere, and outside the photon sphere. However, it turns out that the light cones look much the same in each of these cases, so we will only look at diagrams for one of them. In the diagrams, we have taken $M = 1$, so that the photon sphere is at $r = 3$, and have also chosen $r(P) = 3$. Figures 1 to 6 respectively show the intersection of the light cone of $P$ with $t = 1$, 5, 10, 15, 20 and 30. The solid circle in the centre of each diagram is $r = 2$, so only points outside this circle correspond
to points in the space-time under consideration; points inside the circle do not represent points inside the event horizon—they do not correspond to points of Schwarzschild space at all. The dashed curve gives the intersection of the light cone of \( P \) with the relevant surface of constant time; the dot inside this curve is at \( r = 3, \phi = 0 \). (Note that because of the time-translation and reflection symmetries of Schwarzschild space, we can also think of these as the intersections with \( \{ t = 0 \} \) with the past light cone of \( P \) where \( t(P) = 1, 5, 10 \) etc.)

We see that as \( t \) increases, the region consisting of points causally connected to \( P \) grows, at first like a sphere (as in Minkowski space), but then folding round the event horizon of the black hole. This looks a little like the behaviour of an amoeba enfolding a food particle, but in this case the light cone crosses inside itself and wraps round the black hole repeatedly. We can easily see, however, that a point \( Q \) in \( \{ t = \tau \} \) is causally connected to \( P \) if either it is connected to \( P \) by a null geodesic (i.e. it lies on the null cone of \( P \)) or if the \( P \)-wavefront at time \( \tau \) has non-zero winding number around \( Q \).

In particular, if we are given two points \( P \) and \( Q \) with coordinates \((t(P), r(P), \theta(P), \phi(P))\) and \((t(Q), r(Q), \theta(Q), \phi(Q))\), we can change coordinates so that \( t(P) = 0, \theta(P) = \theta(Q) = \pi/2 \), and \( \phi(P) = 0 \), by subtracting \( t(P) \) from all time coordinates, and carrying out the appropriate rotation. Then all we have to do is construct a diagram of \( \{ t = t(Q) \} \), showing \( Q \) at \((r(Q), \phi(Q))\) and the curve consisting of the equatorial \( P \)-wavefront at time \( t(Q) \), which is the geodesic sphere of radius \( t(Q) - t(P) \) centred on \((r(P), 0)\). If \( Q \) lies inside or on this curve, then the two points are causally connected, otherwise they are not.

We can also see that as \( t \) increases the set of points in each surface of constant time that are causally related to it eventually start to resemble a sphere again, except that there is a region in the centre, consisting of points very near the circle \( r = 2 \) (and, of course, those inside it) which are never causally related to \( P \).

By changing viewpoint a little, we can use the diagrams above to tell us something about gravitational lensing, and what an observer would see who was fortunate enough to fall into a Schwarzschild black hole. (Though falling into a black hole may seem a poor reason for being regarded as fortunate, black holes with no angular momentum are almost certainly so rare as to make the observer at least relatively fortunate.)

First, we must realize that the points at which the light cone of \( P \) cuts the surface \( \{ t = \tau \} \) for negative \( \tau \) may be found in just the same way as for \( \tau > 0 \), by time-symmetry, and consist of precisely those points on \( \{ t = \tau \} \) that are visible to \( P \). (These points comprise a wave-front that comes to focus exactly at \( P \).) So, when the light cone intersects itself, say at \( Q \), what
we have are distinct light rays leaving the same point, that focus together at $P$, and provide him with separate images leaving $Q$ at the same time. In fact, when our diagrams show a simple intersection, we must rotate the diagram to get the full picture. So, there are a circle’s worth of light rays all leaving a point which focus together at $P$, providing him with a ring-shaped image of that point.

There are a couple of points worth noting here:

- This Einstein ring image is not only formed for points on the far side of the black hole from $P$, but also at the near side.

- From this analysis we only see those gravitational lens effects that focus light rays with an equal travel time—hence there is no interference to consider, and focussing only happens with light rays leaving points on the radial line containing $P$.

With a little effort one can use the pictures we have obtained to obtain some information about the images of objects not on the same radial line as the observer; if some object is at a fixed coordinate position, one can see when the light cone of an observer at some fixed coordinate position cuts the world-line of this object, and reconstruct the image. However, for this kind of detail it is probably simpler to use the approach found in Ohanian [9].

We can also consider what happens to an observer falling into (or simply making a very near approach to the event horizon of) such a black hole. As he approaches the event horizon, he crosses the light cone of a given point many times, since it wraps round the event horizon over and over again. So, what an observer sees as he falls in consists of a history of external events, but repeated many times. A more detailed discussion of what an observer near an event horizon sees can be found in Schastok et al. [10].

Finally, we should note that this approach to visualizing the light cones of a point will only work in a space time where the metric is conformally of the type

$$-dt^2 + G_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta$$

where $\alpha$ and $\beta$ run from 1 to 3, and $G_{\alpha\beta}$ is time independent and radially symmetric. Schwarzschild is the only such solution of the Einstein equations corresponding to a vacuum, but there may be interesting effects in metrics in universes containing matter that do not appear in the Schwarzschild case.

11 Acknowledgements

I would like to thank the anonymous referees, who pointed out numerous shortcomings of exposition in an earlier version of this paper, for their con-
References


