Clairaut’s Theorem in Minkowski Space

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Abstract
Clairaut’s theorem gives a well-known characterization of geodesics on a surface of revolution in Euclidean space. Here, we find an analogous result in three-dimensional Minkowski space, and interpret it in the context of low dimensional cosmological models.

1 Euclidean Geometry

We begin by recalling the situation in Euclidean space, the better to see how closely the situation in Minkowski space parallels this one.

Let Σ be a surface of revolution, obtained by rotating the curve \((x = \rho(u), y = 0, z = h(u))\) about the z-axis, where we assume that \(\rho > 0\) and that \(\rho'(u)^2 + h'(0)^2 = 1\). Then Σ is parameterized by

\[ x(u, v) = \begin{bmatrix} \rho(u) \cos(v) \\ \rho(u) \sin(v) \\ h(u) \end{bmatrix} \]

and has first fundamental form

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & \rho(u)^2 \end{bmatrix}. \]

We also have

\[ x_u = \begin{bmatrix} \rho'(u) \cos(v) \\ \rho'(u) \sin(v) \\ h'(u) \end{bmatrix} = n_u \]

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the unit vector pointing along meridians of Σ and
\[ \mathbf{x}_v = \begin{bmatrix} -\rho(u) \sin(v) \\ \rho(u) \cos(v) \\ 0 \end{bmatrix} = \rho \begin{bmatrix} -\sin(v) \\ \cos(v) \\ 0 \end{bmatrix} = \rho \mathbf{n}_v \]
where \( \mathbf{n}_v \) is the unit vector pointing along parallels of Σ.

Now, let \( \gamma(t) \) be a geodesic on Σ, given by \( u(s) \), and \( v(s) \), so that
\[ \gamma(s) = \begin{bmatrix} \rho(u(s)) \cos(v(s)) \\ \rho(u(s)) \sin(v(s)) \\ h(u(s)) \end{bmatrix}. \]
From the first fundamental form, we have the Lagrangian
\[ \dot{u}^2 + \rho^2 \dot{v}^2 \]
and so the Euler-Lagrange equations, whose solutions are arc-length parameterised geodesics, are
\[ \ddot{u} = \rho \rho' \dot{v}^2 \]
\[ \frac{d}{dt} (\rho^2 \dot{v}) = 0. \]
But now, we also have
\[ \dot{\gamma} = \dot{u} \mathbf{x}_u + \dot{v} \mathbf{x}_v \]
\[ = \dot{u} \mathbf{n}_u + \rho \dot{v} \mathbf{n}_v \]
\[ = \mathbf{n}_u \cos \theta + \mathbf{n}_v \sin \theta \]
where \( \theta \) is the angle between \( \dot{\gamma} \) and a meridian.

Since \( \rho \neq 0 \), we now immediately the second Euler-Lagrange equation is equivalent to \( \rho^2 \sin \theta \) being a constant along \( \gamma \).

Conversely, suppose that \( \gamma \) is a geodesic, with \( \dot{v} \neq 0 \). Then since
\[ u^2 + \rho^2 \dot{v}^2 = 1 \]
we have
\[ \ddot{u} + \rho \rho' \dot{v}^2 + \rho^2 \ddot{v} = 0 \]
and substituting this into the second Euler-Lagrange equation,
\[ 2 \rho \rho' \dot{u} \dot{v} + \rho^2 \ddot{v} = 0, \]
yields
\[ \ddot{u} = \rho \rho' \dot{v}^2, \]
which is the first Euler-Lagrange equation.

This established Clairaut’s theorem, and we observe in passing that all meridians are geodesics.
2 Clairaut’s Theorem in Minkowski Space

We now consider the situation of a curve in Minkowski space, which we take to have the usual coordinates \((x, y, t)\) with metric

\[ ds^2 = dx^2 + dy^2 - dt^2, \]

which is rotated about the \(t\)-axis.

So let the curve be given by \(x = \rho(u) > 0, \ y = 0, \ t = h(u),\) and we assume that \(\rho'(u)^2 - h'(u)^2 = -1,\) so that the curve is timelike, and parameterised by proper time.

We then find the the surface \(\Sigma\) is parameterised by

\[ x(u, v) = \begin{bmatrix} \rho(u) \cos(v) \\
\rho(u) \sin(v) \\
h(u) \end{bmatrix} \]

giving

\[ x_u = \begin{bmatrix} \rho'(u) \cos(v) \\
\rho'(u) \sin(v) \\
h'(u) \end{bmatrix} \quad \text{and} \quad x_v = \begin{bmatrix} -\rho(u) \sin(v) \\
\rho(u) \cos(v) \\
0 \end{bmatrix}, \]

and resulting in the first fundamental form

\[ I = \begin{bmatrix} x_u & x_u & x_u & x_v & x_v & x_v \\
x_u & x_u & x_v & x_v & x_u & x_v \end{bmatrix} = \begin{bmatrix} -1 & 0 \\
0 & \rho(u)^2 \end{bmatrix}. \]

which gives a Lorentz metric on \(\Sigma.\)

So we see that \(x_u = n_u\) is a unit timelike vector pointing along the meridians, while \(x_v = \rho n_v,\) where \(n_v\) is a unit spacelike vector pointing along the parallels.

This time the Lagrangian is

\[-\dot{u}^2 + \rho^2 \dot{v}^2\]

giving Euler-Lagrange equations

\[\ddot{u} = -\rho \rho' \dot{v}^2,\]

\[\frac{d}{dt} \left(\rho^2 \dot{v}\right) = 0.\]

Now let \(\gamma\) be a timelike geodesic on \(\Sigma,\) again given by \(u(s), v(s).\) Then as before we have

\[\gamma = \dot{u} x_u + \dot{v} x_v,\]

\[= \dot{u} n_u + \rho \dot{v} n_v.\]
In the Minkowski setting, however, this gives
\[ \dot{\gamma} = n_u \cosh \theta + n_v \sinh \theta \]
where \( \theta \) is now the hyperbolic angle between \( \dot{\gamma} \) and \( n_u \), i.e. between \( \dot{\gamma} \) and a meridian.

We then see that the second Euler-Lagrange equation is equivalent to \( \rho \sinh \theta \) being constant.

Conversely, let \( \gamma \) be a proper-time parameterized curve such that \( \rho \sinh \theta = \rho^2 \dot{v} \) is constant, and \( \dot{u} \neq 0 \). We then have
\[ \dot{u}^2 - \rho^2 \dot{v}^2 = 1 \quad \text{and} \quad \rho^2 \dot{v} = \text{constant} \]
Differentiating this gives
\[ \ddot{u} = \rho \rho' \dot{v}^2 \]
Multiplying the second equation by \( \dot{v} \) and substituting into the first gives
\[ \ddot{u} + \rho \rho' \dot{v}^2 = 0 \]
and since \( \dot{u} \neq 0 \) we have
\[ \ddot{u} = -\rho \rho' \dot{v}^2 \]
which is the second Euler-Lagrange equation. It follows that \( \gamma \) is a timelike geodesic.

We thus see that Clairaut’s theorem has a Minkowski space analogue with \( \rho \sinh \theta \) replacing \( \rho \sin \theta \) as the quantity conserved along a timelike geodesic.

As before, we can immediately deduce that all meridians are geodesics.

In addition, we see that for small values of \( \theta \), the geodesics will be close to those for the Euclidean case.

Null curves on the surface are automatically geodesics. Given by

### 3 Motion in a Cosmological Model

We now turn our attention to the closed FRW cosmological models, whose metric is given by
\[ ds^2 = a^2(t) d\Omega^2 - dt^2 \]
where \( d\Omega^2 \) is the metric on the standard round sphere of radius 1.

In particular, we might ask what is the motion of a freely falling particle in such a cosmology. We can immediately see that by an appropriate choice of
coordinates, we are looking at the timelike geodesics on a Lorentz manifold equipped with the coordinates \( v \in [0, 2\pi) \) (where we identify \( v = 0 \) with \( v = 2\pi \)) and \( t \), with metric
\[
ds^2 = a^2(t)dv^2 - dt^2,
\]
so we seek a surface of revolution in three dimensional Minkowski space with this metric.

Note that the most naive guess, that we simply take the curve \( x = a(t), y = 0 \) is wrong—although it seems to provide a sphere of the correct radius at each time, the induced metric is incorrect and the surface of revolution is not even Lorentzian if \( |\dot{a}| > 1 \).

Instead, we consider \( x = \rho(u), y = 0, t = h(u) \), and seek \( \rho \) and \( \tau \). Just as before, we find that the first fundamental form is
\[
\begin{bmatrix}
\rho'(u)^2 - h'(u)^2 & 0 \\
0 & \rho(u)^2
\end{bmatrix}.
\]
If we then take \( \rho(u) \) to be \( a(u) \), we require \( h(u) \) such that \( \rho'^2 - h'^2 = -1 \), i.e. \( h'^2 = 1 + \rho'^2 \); clearly this can be solved for any differentiable \( a \).

We then have first fundamental form
\[
\begin{bmatrix}
-1 & 0 \\
0 & \rho(u)^2
\end{bmatrix}
\]
so that the intrinsic metric on the surface of revolution is
\[
a(u)^2dv^2 - du^2
\]
where \( v \) is the angular parameter, and \( u \) is the time coordinate.

### 3.1 de Sitter space

In the flat slicing coordinates, de Sitter space has the form
\[
ds^2 = \cosh^2(t)d\Omega^2 - dt^2
\]
Here we have \( a(u) = \cosh(u) \), so \( h'(u) \) must be \( \sqrt{1 + \sinh^2(u)} = \cosh(u) \) again, and thus \( h(u) = \sinh(u) + C \), and we will take \( C = 0 \) so that the scale factor is at its minimum, when \( t = h(u) \).

Then a slice of de Sitter space is embedded isometrically in three-dimensional Minkowski space as the surface obtained by rotating the curve \( x = \cosh(u), y = 0, t = \sinh(u) \) about the \( t \)-axis, or equivalently the curve given by \( x^2 - t^2 = 1 \), which recovers a shadow of the representation of de Sitter space as a hyperboloid of revolution in five-dimensional Minkowski space.
3.2 Bang-Crunch space

Another interesting solution is given by when the scale factor is a cycloid: in this case the scale factor is given parametrically by

\[ a = 1 - \cos(\tau) \quad t = \tau - \sin(\tau) \]

as \( \tau \) varies from 0 to \( 2\pi \).

It then follows that the path of a freely falling particle is entirely determined by Clairaut’s theorem.